PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCXXXI.

PART I.

LONDON:

PRINTED BY RICHARD TAYLOR, RED LION COURT, FLEET STREET.

MDCCCXXXI.

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the council-books and journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries, till the Forty-seventh Volume: the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion,

27703

as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

CONTENTS.

I. *Observations of the Second Comet of* 1822, *made at Rio de Janeiro. By Lieutenant (now Captain)* William Robertson, *R.N. Communicated in a Letter to Captain* Basil Hall, *F.R.S.* . . . . . page 1

II. *On the performance of Fluid Refracting Telescopes, and on the applicability of this principle of construction to very large instruments. By* Peter Barlow, *Esq. F.R.S. Cor. Mem. Inst. of France, of the Imperial Academy of St. Petersburgh, &c. &c.* 9

III. *Researches in Physical Astronomy. By* John William Lubbock, *Esq. V. P. and Treas. R.S.* 17

IV. *On the Transient Magnetic State of which various Substances are susceptible. By* William Snow Harris, *Esq. Communicated by* Davies Gilbert, *Esq. V.P.R.S.* 67

V. *On the Nature of Negative and of Imaginary Quantities. By* Davies Gilbert, *Esq. V.P.R.S.* 91

VI. *On the probable Electric Origin of all the Phenomena of Terrestrial Magnetism; with an illustrative Experiment. By* Peter Barlow, *Esq. F.R.S. Cor. Mem. Inst. of France, of the Imperial Academy of St. Petersburgh, &c. &c.* 99

VII. *On the Equilibrium of Fluids, and the Figure of a Homogeneous Planet in a Fluid State. By* James Ivory, *A.M. F.R.S. Instit. Reg. Sc. Paris. Corresp. et Reg. Sc. Gottin. Corresp.* 109

VIII. *On a simple Electro-chemical Method of Ascertaining the Presence of different Metals, applied to detect minute quantities of Metallic Poisons. By* EDMUND DAVY, *F.R.S. M.R.I.A. &c. Professor to the Royal Dublin Society* . . . . . page 147

IX. *On the Chemical Action of Atmospheric Electricity. By* ALEXANDER BARRY, *Esq. F.L.S. Communicated by* JOHN GEORGE CHILDREN, *Esq. Sec. R.S., &c. &c.* 165

X. *An Account of Operations carried on for ascertaining the Difference of Level between the River Thames at London Bridge and the Sea; and also for determining the Height above the Level of the Sea, &c. of intermediate Points passed over between Sheerness and London Bridge. By* JOHN AUGUSTUS LLOYD, *Esq. F.R.S. F.R.G.S. & F.S.A.* 167

XI. *On the Variable Intensity of Terrestrial Magnetism, and the Influence of the Aurora Borealis upon it. By* ROBERT WERE FOX. *Communicated by* DAVIES GILBERT, *Esq. M.P. V.P.R.S.* 199

XII. *Description of a Graphical Registrer of Tides and Winds. By* HENRY R. PALMER, *Civil Engineer. Communicated by* JOHN WILLIAM LUBBOCK, *Esq. V.P. and Treas. R.S.* 209

XIII. *On the Errors in the Course of Vessels, occasioned by Local Attraction; with some Remarks on the recent Loss of His Majesty's ship Thetis. By* PETER BARLOW, *Esq. F.R.S. Cor. Mem. of the Inst. France, Imp. Acad. Sciences St. Petersburgh, Acad. Sciences Brussels, &c. &c.* 215

XIV. *On the Meteorological Observations made at the Apartments of the Royal Society, during the Years* 1827, 1828, *and* 1829. *By* J. W. LUBBOCK, *Esq. V.P. and Treas. R.S.* 223

APPENDIX.

Meteorological Journal kept at the Apartments of the Royal Society, by order of the President and Council.

The PRESIDENT and COUNCIL of the ROYAL SOCIETY adjudged the ROYAL MEDALS for the year 1830 as follows:

A Royal Medal to DAVID BREWSTER, LL.D., Fellow of the Royal Society, for his Communications to the Royal Society on the Polarization, and other Properties, of Light.

A Royal Medal to M. BALARD, of Montpellier, for his Discovery of Brome.

PHILOSOPHICAL TRANSACTIONS.

I. *Observations of the Second Comet of* 1822, *made at Rio de Janeiro. By Lieutenant (now Captain)* William Robertson, *R.N. Communicated in a Letter to Captain* Basil Hall, *F.R.S.*

Read June 17, 1830.

I SHALL feel obliged to you to lay before the Royal Society the following observations, which, with the assistance of Lieut. Charles Drinkwater, R.N., I made upon the "second comet of 1822," as it is called. They were made at Rio de Janeiro, when I was Lieutenant of His Majesty's ship Creole, under the orders of Commodore Sir Thomas Hardy; but as the means I had in my possession for making such nice observations were not great, I did not imagine the results could be of much value, till I accidentally gained information of the following remarks in Baron Zach's Journal, vol. vi. page 596.

"Cette comète, comme nous l'avons dit dans notre V^me^ cahier, page 481, n'a été que très peu observée, à cause de son mouvement fort-rapide en déclinaison australe; elle s'est par conséquent bientôt soustraite à nos regards, pour se montrer peut-être avec plus d'éclat aux antipodes. Nous n'avons reçu d'autres observations que celles que nous avons déjà publiées dans notre cahier précédent. Probablement l'orbite de cet astre passager nous restera inconnue pour toujours, à moins que MM. Fallow, Rumker, ou un autre Basil Hall, ne parviennent à l'observer dans l'hémisphere austral. Mais la correspondence astronomique des deux hémispheres n'est pas encore bien établie *."

The above allusion to your observations on the comet which we observed together at Valparaiso in 1821, and which are published in the Philosophical

* Correspondence Astronomique du Baron Zach, vol. vi. p. 595.

Transactions, induces me to address this communication to you, to be laid before the Society, if you consider it worthy of that honour.

M. Pons, it appears, (Zach, vol. vi. p. 385) first discovered this comet on the 31st of May, about two o'clock in the morning. It was then at the distance of $2\frac{1}{2}°$ from the star β Piscium, 5° from γ Aquarii, nearly in $340\frac{1}{3}°$ of $\mathcal{R}$, and in $2\frac{1}{3}°$ north declination. As M. Pons had no instruments ready to observe the comet, it does not appear to have been further noticed till the 8th of June, by Professor Caturegli at Bologna, and two days afterwards by M. Gambart at Marseilles. Owing to its extreme feebleness, however, Baron Zach does not appear to think the observations at Bologna very exact.

The following two Tables of the observations just mentioned are from Zach's Correspondence, vol. vi. p. 482.

1822.	Tems vrai à Bologne.	Ascen. droite de la comète.	Declin. de la comète australe.
	h m	° ′	° ′
Juin 8	15 10	347 39	8 49
10	14 45	351 43	13 28
11	14 44	354 32	16 46
12	14 59	358 25	21 5

The following are the only two observations made by M. Gambart at Marseilles.

1822.	Tems moyen de minuit.	Differ. d'ascen. droite.	N° d'obs.	Differ. de declin.	N° d'obs.	Etoiles companées.
	h m s	° ′ ″		′ ″		
Juin 10	3 3 49	+2 11 9.2	1	−21 38.1	1	ψ^5 du Verseau.
11	2 48 56	−0 20 27.0	4	+30 31.3	4	133 Hor. xxiii. Piazzi.

From these few observations, M. Hulingenstein has deduced the orbit in Schumacher's Astronomische Nachrichten, vol. iv. pp. 533, 534, and which I have copied at the end of this letter. But as the above observations comprehend only a very small portion of the orbit, these elements are susceptible of improvement from observations made during a longer interval.

The following are the whole of the observations which were made by Mr. Drinkwater and myself on this comet.

On the evening of the 18th of June it was first observed, with the naked eye, near the star Canopus, and though it had been in conjunction with the sun on that day, its great southern latitude permitted it to be seen after sunset. Of course we did not know that it had been observed in Europe, and immediately proceeded to determine its position by the only means of which we had command. By means of a reflecting sextant, we took its angular distances from known fixed stars. The faintness of the comet's light, and the uncertainty in estimating its apparent centre, presented considerable difficulties in the employment of this method of observation. It was attempted to diminish the errors which, no doubt, arose from this cause, by taking the angular distances from four stars, and these observations were repeated, as often as the weather permitted, on the succeeding nights. In this manner angular distances were obtained on the evenings of the 18th, 19th, 22nd, 23rd, and 24th of June, 1822; after which the increasing brightness of the moonlight, and the faintness, prevented its being accurately observed with the sextant, and eventually obliterated it altogether.

During the whole of the above period, the comet presented the same appearance as it seems to have done in Europe,—namely, that of a nebulous mass, without either tail or nucleus. I was of course inclined to believe, at first, that these observations would be of little value, as I did not doubt that the comet must have been observed from other places in the southern hemisphere, with more efficient instruments than mine. I hope it may still prove so; but, after repeated inquiries, I have not been able to learn that the comet was seen in any other quarter of that portion of the globe. Nor, indeed, was it until its orbit had been computed from my observations by Mr. Thomas Henderson of Edinburgh, that the comet was suspected to have been one previously observed in Europe. This discovery, it will perhaps be thought, gives a new value to the observations which we made; for though the means used were deficient in that precision which is desirable, the observations, taken in conjunction with those made in Europe, embrace a far greater extent of the comet's orbit than either series do alone. And in the present state of cometary astronomy it is impossible to foretell the value which may one day be assigned to observations which at present appear to have little interest.

The following is a faithful transcript of the original observations as they

were actually written down at the time; without the alteration of a letter or a figure.

Rio de Janeiro, June 18th, 1822, at $6^h\ 30^m$ P.M. Observed a bright orbicular nebula near Canopus. On directing the telescope to it, we find it to have the appearance of a comet. At $6^h\ 40^m$ mean time, the following distances were taken with sextants:

From Canopus . . .	3°	6′	20″
—— Sirius	34	27	10
—— α Hydræ . . .	58	9	20
—— α Crucis . . .	47	58	50

June 19th. The comet appeared fainter than last night. There was a thin haze in the sky. The following observations were taken at $6^h\ 40^m$ P.M.:

From Canopus . . .	11°	33′	30″
—— Sirius	30	3	37
—— α Hydræ . . .	46	2	47
—— α Crucis . . .	44	15	30

June 20th. Thick, rainy weather; comet not seen.

June 21st. Thick, cloudy weather.

June 22nd. Fine, clear moonlight. Observed the comet without a telescope. It is still of a round shape, no tail or nucleus observed when looked at with a telescope. The following angular distances were taken at $7^h\ 0^m$ P.M.:

From Canopus . . .	33°	35′	00″
—— Sirius	33	12	00
—— α Hydræ . . .	25	9	45
—— α Crucis . . .	44	36	25

June 23rd. Clear weather. The following angular distances were taken at $6^h\ 34^m$ P.M.:

From Canopus . . .	37°	29′	20″
—— Sirius	35	15	45
—— α Hydræ . . .	21	38	50
—— α Crucis . . .	45	13	10

June 24th. Clear weather; moonlight. The following distances were taken at $6^h\ 30^m$ P.M.:

From α Hydræ	18° 57′ 25″
—— α Crucis	46 37 30

June 25th. Saw the comet; but owing to the clear moonlight, it was too faint to be observed with the sextant.

June 26th. Dark cloudy weather, with rain and thunder.

June 27th. Rainy weather. In the evening, fine weather; comet not seen.

June 28th. Cloudy evening.

June 29th. Fine clear moonlight; could not discover the comet.

On my attention being called to this comet during the last year, more than seven years after I had observed it, I placed the above observations in the hands of Mr. Thomas Henderson, of Edinburgh, who has furnished me with the following remarks, which I transcribe verbatim, and request you will communicate to the Royal Society in the same manner.

"From the observations," says Mr. Henderson, "made at Rio de Janeiro, by Captain Robertson and Lieutenant Drinkwater, on the second comet of 1822, I have obtained the following position of that comet referred to the ecliptic, and cleared of the effect of refraction, but not of parallax, aberration, nutation, or precession. Those positions have been adopted which represent the observed angular distances with the minimum of error, as found by the method of least squares.

Mean solar time at Rio de Janeiro. June 1822.	Apparent longitude.	Apparent latitude.
d h m	° ′ ″	° ′ ″
18 6 40	93 39 26	73 51 6 South
19 6 40	125 15 42	66 42 19
22 7 0	147 5 5	47 30 47
23 6 34	149 31 36	43 49 20
24 6 30	150 48 47	40 39 58

"The errors of observation, on the differences between the observed and computed angular distances, do not exceed five minutes of space, except on the

23rd, when, in one observation, the error amounts to eleven minutes; for which reason the position of that day is not employed in computing the orbit.

"The following elements of the comet's parabolic orbit have been obtained by OLBERS' method of computation, founded upon the observations of June 19th, 22nd, and 24th.

Time of perihelion passage, mean solar time at Greenwich, 1822, July 15.651.

Longitude of the perihelion	220° 19′ 49″
Inclination of the orbit	35 36 0
Longitude of the ascending node . . .	98 14 47
Logarithm of perihelion distance . . .	9.92879

Motion retrograde.

"The following are the errors of the places computed from these elements, or the corrections to be applied to the computed places, in order to obtain those which were observed.

	Longitude.	Latitude.
June 18	− 7″	+ 1′
19	+ 5	+ 1
22	0	− 1
23	+ 15	+ 5
24	0	+ 1

"The greatest error is on the 23rd; the observations of which day, for the reasons already stated, are supposed not to be so exact as those of the other days. The other errors, it may be remarked, are not greater than what might have been expected from the uncertainty of the observations, and great latitude of the comet, when the errors in longitude are apparently much increased, from being reckoned upon a small circle.

"On comparing the foregoing elements, computed from Captain ROBERTSON and Lieutenant DRINKWATER'S observations, with those deduced by M. HULINGENSTEIN from the observations made in Europe, referred to at page 2, it will be seen that the differences between them are wonderfully small, considering the different instruments used by the observers in the two hemispheres.

"The elements, placed side by side, stand thus:

	By M. Hellingenstein's computation from observations in Europe.	By the computations from Captain Roberton's observations at Rio de Janeiro.
Time of perihelion passage,	July 16.03925	July 15.651
	Mean Solar Time at Marseilles.	Mean Solar Time at Greenwich.
Longitude of the perihelion	219° 53′ 48″	220° 19′ 49″
Inclination of the orbit	37 43 4	35 36 0
Longitude of the ascending node	97 51 23	98 14 47
Logarithm of perihelion distance	9.92743	9.92879

"Perhaps more correct elements might be obtained from a comparison of all the observations, European as well as South American, were it deemed of sufficient importance to undergo the requisite labour. But without entering into such an investigation, enough has been already stated to show that the instruments and other means in the possession of every naval officer, are sufficient to enable him to determine, with considerable accuracy, the orbit of any comet which is not too faint for being observed with the usual reflecting instruments used at sea."

Before concluding this communication, it may not be improper to mention that about the same time that we were making the observations above detailed, on the "second comet of 1822," we were fortunate enough to see the celebrated comet of Encke, but it had not sufficient light to enable us to observe it in the same manner that we did the other. We were therefore obliged to content ourselves with observing it through an ordinary telescope. But, as it does not appear that on this return of Encke's comet to the neighbourhood of the earth, it was seen in any other part of the world, except at Paramatta, the following notes of what we saw of it at Rio de Janeiro, may not be altogether uninteresting, though probably of little or no value to astronomers.

Memorandum of Encke's comet seen at Rio de Janeiro in 1822.

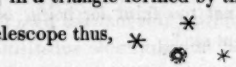
June 7th. At $6^h\ 30^m$ P.M. Observed the comet calculated by Professor Encke, in the constellation Gemini. It was only seen through a telescope, and appeared like a faint nebula of a round form. There were two stars of the 5th

or 6th magnitude near it, with which it formed a right-angled triangle; the right angle at the northernmost of the two stars, and the comet to the westward.

June 10th. Observed ENCKE's comet after sunset. It has increased its Æ. The stars seen along with it on the 7th are not now in the field of view of the telescope at the same time with the comet.

June 12th. Observed ENCKE's comet after sunset. It was very faint. No stars that we have in our catalogues (which are very limited) in the field of the telescope.

June 13th. Observed ENCKE's comet forming an angle of about 100° at β Canis Minoris, with Procyon; at about once and a quarter the distance from β Canis Minoris, that β is from Procyon. It is not brighter than when it was first seen.

June 17th. ENCKE's comet again seen. A line drawn from Sirius to β Canis Minoris cuts a star of the 3rd or 4th magnitude: about $\frac{1}{3}$th of the distance from that star to Procyon, was the comet, in a triangle formed by three stars of the 5th or 6th magnitude, seen by the telescope thus,  , the Æ being about 103°, and declination 5° north, and it has still the same nebulous, orbicular appearance as when first seen.

June 18th. Saw ENCKE's comet after sunset—very faint. It had increased its Æ considerably since last night, from the small stars seen last night in the field of the telescope.

June 19th. Hazy, and the direction of the comet not seen.

June 20th and 21st. Thick weather; comet not seen.

June 22nd. Fine clear moonlight; ENCKE's comet could not be made out, nor was it again seen.

If you think any of these observations likely to interest the Royal Society, I request you will do me the honour to present them.

II. *On the performance of Fluid Refracting Telescopes, and on the applicability of this principle of construction to very large instruments.* By Peter Barlow, *Esq. F.R.S. Cor. Mem. Inst. of France, of the Imperial Academy of St. Petersburgh, &c. &c.*

Read December 9, 1830.

In the Philosophical Transactions for 1827, a paper of mine was published containing an account of a series of experiments which I had carried on with Messrs. W. and T. Gilbert on the curvature of object-glasses for telescopes. In the course of these experiments, I saw so much the difficulty which opticians experience in obtaining large pieces of good flint-glass, that I turned my attention to supplying this material by a fluid. Having, after several attempts, at length found an admirable substitute in sulphuret of carbon, I wrote a short account of my intended construction, addressed to His present Majesty, at that time Lord High Admiral, and, as such, President of the Board of Longitude, soliciting from that Board assistance in carrying forward my experiments. Having obtained this aid, the result of my first trial was the construction of an eight-inch fluid telescope, at that time the largest refractor in this country. A description of this instrument is given in the Philosophical Transactions for 1829, and some objects are pointed out which had been selected as tests of its performance.

I have however since had more time and better means of testing the instrument; first, through the kindness of Mr. Herschel, who pointed out to me several objects that he had observed with his new twenty-inch speculum; and secondly, by direct observations on the same objects in Sir James South's new twenty-feet refractor, and in my own telescope. A few of these, which serve to mark distinctly the progress I have made, are given below; but I will first state two or three of my own observations, which, I conceive, tend also to the same object.

In the paper last referred to, I have stated my observation on γ Persei,

marked as double in South and Herschel's catalogue, with a small star at a greater distance; this star is seen distinctly sextuple in my telescope. These stars I had the satisfaction of showing to M. Struve in his recent visit to England, and I have since seen them in Sir James South's telescope. Another good test of the light of my telescope is found in σ Orionis, marked in the above catalogue as two distinct sets of stars, each triple; whereas, in my telescope, both sets are quadruple, with a double star, or rather two very fine stars between them; the fourth star in the bright set, is a remarkably fine brilliant point, very near to the principal star, and in the same line as the nearest of the original small stars, on the opposite side, so that the three are in one line; or more accurately, the line joining the two small stars touches the margin of the bright star. I might mention several other cases of fine double stars which I have discovered, but I select the above because it is evident that both objects have been well examined with fine instruments, and that the stars I have mentioned had, notwithstanding, escaped detection.

Of the tests furnished me by Mr. Herschel I shall only select two, one of which in particular serves to point out in a very precise manner the limit of power of my telescope. This is the star β Capricorni, which, in the finder, is a coarse double star of about 3′; but between these two stars, nearly in the middle, but a little below the line of junction, is a very fine double star, discovered by Mr. Herschel, and which he considers a very severe test; he says indeed that he requires no other, of the light of a telescope. This star I can see, and, under favourable circumstances, distinctly; but still I have not sufficient command of it to see it double. We have thus the exact limit defined at which the light of this splendid instrument surpasses that of my telescope. The other object to which I have alluded is φ Virginis: this he considers a very easy double star, although it had before escaped detection; it is however rather close. This star I could see very distinctly one evening (June 4th), the moon being very bright and full, on the meridian, and within an hour of the star. I mention this object because it requires a certain degree of defining power; in point of light it involves no difficulty. Mr. Herschel could see it when his aperture was reduced to six inches.

Amongst the objects which I have seen in Sir James South's twenty-feet, there is also one in particular which forms a good test of the relative power of

his instrument and mine; this is MESSIER's twenty-second nebula. This object, which in a good 3½-inch refractor has only the appearance of a white cloud, I saw in the above instrument resolved into an immense number of brilliant small stars. In my telescope also it is resolved into apparently as great a number of stars, but the full power of the instrument is exerted, and still the resolution seems scarcely complete. In fact, my instrument appears to labour to effect what seems to be quite within the power of the other.

I wish particularly to direct attention to this object and that of β Capricorni, because, where the same object can be seen without any very apparent difference in two instruments, or where it can be only seen in one, the great test of comparison is lost; but in those I have mentioned, the exact limit of power is defined.

Amongst the objects I have examined in Sir JAMES SOUTH's telescope, and repeated in mine, for the purpose of comparing the defining powers of the two instruments, were the planets Jupiter and Mars. These were both more sharply defined in Sir JAMES SOUTH's than in my telescope, but the superiority was by no means so great as I had expected. I fortunately saw the shadow of Jupiter's third satellite pass over the disc, on the 8th of August, at Kensington, and it exhibited a fine black round spot, extremely well defined; and on the 13th of the same month, I witnessed precisely the same phenomenon in mine, and, as far as the definition of the shadow was concerned, with an effect in which I could not distinguish an inferiority; it had the appearance of a small black wafer on a sheet of white paper; still, however, the edge of the planet was certainly sharper in the twenty-feet. Both evenings were amongst the finest this climate affords, and the powers employed as nearly as possible equal; viz. about 260 and 450 in both instruments. We also, at Kensington, observed Mars with various powers from 260 to 1400, and it carried 1200 well; with this the white spot near its south pole was seen beautifully distinct, as also a long dark spot on its apparent eastern limb. The bright spot at its south pole I saw also remarkably well defined in my own telescope on the 13th, and a dark spot on its disc very distinct; but it occupied the centre of its disc. The highest power however I used was 500, and it was probably best seen with 260.

There can be no question of the superior defining power of the twenty-feet,

and the light is of course also greater; still, however, when I consider that I have been comparing with two telescopes, one of twenty inches aperture, and the other of twelve inches, and each of them twenty feet in focal length, or nearly, while my telescope is barely eight inches aperture, and only twelve feet in length, I cannot but consider the comparison as highly satisfactory.

In addition to the above observations, which have been certainly highly gratifying to myself, I have also had the honour of showing the instrument to many persons, both Englishmen and foreigners well acquainted with astronomy, and in every instance the practicability of the principle of construction has been admitted; a point by no means generally granted when the suggestion was first advanced.

Other obstacles also, independent of the arrangement of the lenses, were foreseen, which time is gradually dissipating; such as the difficulty of permanently securing the fluid, and then, admitting this to be effected, the probability of a decomposition of the glass by the fluid, &c. &c. I have, however, now the satisfaction to state, that the lens of my 3-inch telescope, filled August 5th 1827, continues in precisely its original state, no perceptible change having yet taken place in either the quantity or quality of the fluid, or in the transparency of the glass.

As far as the above observations and remarks extend, therefore, it appears that the essential properties of the flint lens are supplied by the fluid. I beg now to state a few particulars in which the sulphuret of carbon has advantages which the glass has not:—these are, first, that in consequence of the very high dispersive power of this fluid, the correcting lens is placed so far behind the principal plate or crown lens, as to require to be only one half as much in diameter; a highly important consideration in the construction of a very large telescope.

Secondly, the combination is such as to give a focal power one and a half times the length of the tube, or, which is the same, the telescope may be reduced to two thirds the length of a glass telescope of the usual kind, without incurring a greater amount of spherical aberration in the front lens.

Of the latter advantage, however, I have not ventured fully to avail myself in my 8-inch, because, as I knew the general opinion was against the success of the experiment, I was fearful of failing in the beginning by attempting too much.

I have therefore made the length twelve feet, to an aperture of eight inches, which, although shorter than opticians would choose to work in the usual achromatic, is not so short as this principle of construction would admit, and which in any new case I should not hesitate to adopt. Indeed, according to the form of construction I am now about to propose, a telescope of two feet aperture and twenty-four feet in length would not have more spherical aberration to contend with, than a telescope of the usual construction of six inches aperture and twelve feet length, which is fully within the range of the usual practice; at the same time I will not undertake to say that I could on so large a scale confine the length to twelve times the aperture, although I should certainly attempt it in the first instance. But if the length extended to even fifteen or eighteen times the aperture, I have little doubt of making the instrument manageable by one person, by adequate mechanical arrangements, and of producing a telescope which would as much exceed the most powerful telescopes of the present day, as these exceed the refractors of highest repute at the close of the last century.

Whether such an instrument will be undertaken at present, depends upon circumstances which I cannot command. I can only say, that if such a construction were entrusted to my direction, no exertion should be wanted on my part, to render it complete and worthy of the present state of English science. At all events I cannot doubt that the spirit of scientific enterprise will lead ultimately to the attempt; and in order to facilitate the accomplishment of it, as far as lies in my power, I have in the following pages described the nature of the arrangements which in my opinion would most contribute to success.

In my former paper I have given a formula expressing the relations between the length, foci, and distances of the lenses, and have remarked upon the almost infinite variety of forms to which it leads; some of them, I have stated, would probably be found in practice preferable to others, although they are all equally correct in theory. Of these cases, some have since suggested themselves to me; and others will also probably be detected, by a due examination of the formula and tables, which Professor Littrow, of Berlin, has recently presented to the Astronomical Society, relative to this form of telescope; with tables of curvatures, both direct from the formulæ of Euler (reduced to the

case of open lenses), as also indirectly from principles of his own. I have not as yet had an opportunity of examining these cases, but, from the well-known ingenuity of their author, I cannot doubt of finding in this memoir many useful suggestions.

The great change, however, which I propose to make in the construction of this giant telescope, is to have two front lenses, which will be attended with advantages not involved in the above considerations. At present, in consequence of the diameter of the fluid lens being only half that of the front lens, it is difficult to get a sufficient quantity of spherical aberration in the former, to correct that of the latter;—for although we give to the plate lens the curvature requisite for reducing its aberration to a minimum, yet the fluid lens is obliged to be made considerably concavo-convex (a form not to be used when it can be avoided), in order to produce a sufficient aberration in the fluid to correct it. Moreover I have hitherto employed parallel meniscus cheeks to contain the fluid, which present a practical difficulty, if not a positive impediment, to good centering. This will be seen immediately when we consider that when a lens is double-concave or convex, and also when it is concavo-convex, if the radii of curvature are very unequal, the centering may always be effected: for the line joining the centres of the two spheres, or this line produced in the latter case, must pass through the lens, and indicate its true centre: but when the lens has parallel surfaces, or the radii equal, if the two spherical centres be not coincident from the tool itself (a very improbable case) the line which joins them can never cut the lens, and consequently it can have no true centre. All these evils will, however, be avoided in the proposed application of two front lenses, which, by being placed each in what opticians call their best position, will at once reduce the spherical aberration of the front lens to about one third of its present amount, and thereby enable us to correct it by the fluid lens without adopting the distorted form rendered necessary under present circumstances.

Another important consideration is also involved in this form, relative to the facility it affords of obtaining the plate-glass. If the front lens were single, the thickness would be such as would require the glass to be made specifically for the purpose, and of course all the delay and expense of previous experiments would be incurred; whereas, by dividing the whole amount of curvature

between the two, the usual thickness of plate-glass, as at present manufactured, would be sufficient, and we might have the selection from large stores of the best glass at a trifling expense; and as to that of the correcting fluid, or substitute for the flint-glass, it is so very inconsiderable as not to deserve being mentioned; although, if it were possible to obtain a piece of flint-glass large enough for such a purpose, scarcely any price, however great, would be thought exorbitant. In the instrument proposed, nearly the whole expense would be the workmanship, and I must think it very inconsiderable in comparison with the magnitude and importance of the undertaking.

I had intended to have concluded this paper by giving the curvature, foci, &c. which I have computed; but as they are merely suppositious, as far as they are dependent on the index and dispersion of the front glass, it is perhaps better to withhold them. The only object I had in making them was to form some idea of the requisite curvature, thickness of glass, &c. They can only of course be permanently made after the plate-glass has been selected.

III. *Researches in Physical Astronomy.* By JOHN WILLIAM LUBBOCK, *Esq.*, *V.P. and Treas. R.S.*

Read December 9, 1830.

IN last April I had the honour of presenting to the Society a paper containing expressions for the variations of the elliptic constants in the theory of the motions of the planets. The stability of the solar system is established by means of these expressions, if the planets move in a space absolutely devoid of any resistance *, for it results from their form that however far the approximation be carried, the eccentricity, the major axis, and the tangent of the inclination of the orbit to a fixed plane, contain only periodic inequalities, each of the three other constants, namely, the longitude of the node, the longi-

* When the body moves in a medium which resists according to any power of the velocity, the contrary obtains, the major axis and eccentricity acquiring a term which varies with the time, while the longitude of the perihelion and longitude of the epoch have only periodic inequalities. This results from the equations given in the former part of this paper, Phil. Trans. Part II. 1830, page 340.

$$\mathrm{d}a = -2c\,a\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n+1}{2}}\frac{(1-e\cos v)}{n}\,\mathrm{d}v$$

$$\mathrm{d}e = -2c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}}\left(\frac{1-e^2}{2}\right)\cos\mathrm{d}v$$

$$e\,\mathrm{d}\varpi = -2c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}}\frac{\sqrt{1-e^2}}{n}\sin v\,\mathrm{d}v$$

$$\mathrm{d}\varepsilon - \mathrm{d}\varpi = 2c\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}}\left\{\frac{1-e^2\cos v}{e}\right\}\mathrm{d}v$$

$$\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n+1}{2}} = 1+(n+1)\,e\cos v+(n+1)^2e^2\cos^2 v$$

$$\left\{\frac{1+e\cos v}{1-e\cos v}\right\}^{\frac{n-1}{2}} = 1+(n-1)\,e\cos v+(n-1)^2e^2\cos^2 v$$

tude of the perihelion, and the longitude of the epoch, contains a term which varies with the time, and hence the line of apsides and the line of nodes revolve continually in space. The stability of the system may therefore be inferred, which would not be the case if the eccentricity, the major axis, or the tangent of the inclination of the orbit to a fixed plane contained a term varying with the time, however slowly.

The problem of the precession of the equinoxes admits of a similar solution; of the six constants which determine the position of the revolving body, and the axis of instantaneous rotation at any moment, three have only periodic inequalities, while each of the other three has a term which varies with the time. From the manner in which these constants enter into the results, the equilibrium of the system may be inferred to be stable, as in the former case. Of the constants in the latter problem, the mean angular velocity of rotation

---

$$\mathrm{d}a = -\frac{2ca}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\{1 + ne\cos v + n(n+1)e^2\cos^2 v + \&c.\}\,\mathrm{d}v$$

$$= -\frac{2ca}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{1 + \frac{n(n+1)}{2}e^2 + ne\cos v + \frac{n(n+1)}{2}e^2\cos^2 v + \&c.\right\}\mathrm{d}v$$

$$\mathrm{d}e = -\frac{2c}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\{\cos v + (n+1)e\cos^2 v + (n+1)^2e^2\cos^3 v + \&c.\}(1-e^2)\,\mathrm{d}v$$

$$= -\frac{2c}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{n+1}{2}e + \left(1 + 3\frac{(n+1)^2}{4}\right)\cos v + \frac{(n+1)}{2}e\cos 2v + \frac{(n+1)^2}{4}\cos 3v + \&c.\right\}(1-e^2)\,\mathrm{d}v$$

neglecting the terms which are periodic

$$\mathrm{d}a = -\frac{2ca}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{1 + \frac{n(n+1)}{2}e^2 + \&c.\right\}\mathrm{d}v$$

$$\mathrm{d}e = -\frac{2c}{n}\left(\frac{\mu}{a}\right)^{\frac{n-1}{2}}\left\{\frac{(n+1)}{2}e + \&c.\right\}(1-e^2)\,\mathrm{d}v$$

The major axis decreases perpetually, the eccentricity diminishes perpetually until it reaches zero, while the perihelion retains the same *mean position*, and the longitude of the epoch the same *mean value*. I stated inadvertently in the former part of this paper, p. 340, that the variations of the eccentricity are all periodical.

may be considered analogous to the mean motion of a planet, or its major axis; the geographical longitude, and the cosine of the geographical latitude of the pole of the axis of instantaneous rotation, to the longitude of the perihelion and the eccentricity; the longitude of the first point of Aries and the obliquity of the ecliptic, to the longitude of the node and the inclination of the orbit to a fixed plane; and the longitude of a given line in the body revolving, passing through its centre of gravity, to the longitude of the epoch. By the stability of the system I mean that the pole of the axis of rotation has always nearly the same geographical latitude, and that the angular velocity of rotation, and the obliquity of the ecliptic vary within small limits, and periodically. These questions are considered in the paper I now have the honour of submitting to the Society. It remains to investigate the effect which is produced by the action of a resisting medium; in this case the latitude of the pole of the axis of rotation, the obliquity of the ecliptic, and the angular velocity of rotation might vary considerably, although slowly, and the climates undergo a considerable change.

The co-efficients of the terms in the development of R, multiplied by the squares and products of the eccentricities, are susceptible of very great simplification, in consequence of the equations of condition which obtain between the quantities of which the general symbol is b. I have now given the development of R, as far as the terms depending upon the squares and products of the eccentricities, in its simplest form. See p. 30.

I have also given methods of obtaining the inequalities of the radius vector, of longitude, and of latitude in the planetary theory. The expressions in this paper differ in form from those of LAPLACE, but their identity may be shown by means of equations of condition which obtain between some of the quantities involved.

I have taken as a numerical example, the calculation of the co-efficients of some of the inequalities in the theory of Jupiter, disturbed by Saturn.

On the Precession of the Equinoxes.

Let O be the origin of the co-ordinate axes, coinciding with some point in the interior of the mass M.

Let x, y, z be the co-ordinates of any element $\mathrm{d}m$ parallel to three rectangular

axes Ox, Oy, Oz, fixed in space, x_ρ, y_ρ, z_ρ the co-ordinates of the same element parallel to three other rectangular axes Ox_ρ, Oy_ρ, Oz_ρ fixed in the mass M and revolving with it. Let the line NON′ be the intersection of the plane $x_\rho y_\rho$ with the plane xy,

Let the angle $NOx = \psi$, $NOx_\rho = \varphi$, and the inclination of the plane $x_\rho y_\rho$ upon $xy = \theta$.

$$x_\rho = x(\cos\theta \sin\psi \sin\varphi + \cos\psi \cos\varphi) + y(\cos\theta \cos\psi \sin\varphi - \sin\psi \cos\varphi) - z \sin\theta \sin\varphi$$
$$y_\rho = x(\cos\theta \sin\psi \cos\phi - \cos\psi \sin\phi) + y(\cos\theta \cos\psi \cos\varphi + \sin\psi \sin\varphi) - z \sin\theta \cos\varphi$$
$$z_\rho = x \sin\theta \sin\psi + y \sin\theta \cos\psi + z \cos\theta$$

Let X_ρ, Y_ρ, Z_ρ be the accelerating forces which act upon the element dm in the direction of the axes Ox_ρ, Oy_ρ, Oz_ρ,

$$\int(y_\rho^2 + z_\rho^2)\,dm = A, \qquad \int(z_\rho^2 + x_\rho^2)\,dm = B, \qquad \int(x_\rho^2 + y_\rho^2)\,dm = C$$

$$p\,dt = \sin\varphi \sin\theta\,d\psi - \cos\varphi\,d\theta$$
$$q\,dt = \cos\varphi \sin\theta\,d\psi + \sin\varphi\,d\theta$$
$$r\,dt = d\varphi - \cos\theta\,d\psi$$

$$C\,dr + (B - A)pq\,dt = dt\int(x_\rho Y_\rho - y_\rho X_\rho)\,dm$$

$$B\,dq + (A - C)rp\,dt = dt\int(z_\rho X_\rho - x_\rho Z_\rho)\,dm$$

$$A\,dp + (C - B)qr\,dt = dt\int(y_\rho Z_\rho - z_\rho Y_\rho)\,dm$$

If the axis of instantaneous rotation coincides with the line OI at any instant

$$\cos IOx_\rho = \frac{p}{\sqrt{p^2 + q^2 + r^2}}$$

$$\cos IOy_\rho = \frac{q}{\sqrt{p^2 + q^2 + r^2}}$$

$$\cos IOz_\rho = \frac{r}{\sqrt{p^2 + q^2 + r^2}}$$

$$\sin IOz_\rho = \frac{\sqrt{p^2 + q^2}}{\sqrt{p^2 + q^2 + r^2}}$$

If z_ρI L be a great circle cutting the plane $x_\rho y_\rho$ in L,

$$\cos x_\rho OL = \frac{\cos IOz_\rho}{\sin IOz_\rho} = \frac{p}{\sqrt{p^2 + q^2}}$$

If the accelerating forces $X, Y, Z = 0$ and $B = A$, the integrals of the preceding equations are

$$r = n, \qquad p = c\cos\frac{C - A}{A}(nt + \nu), \qquad q = c\sin\frac{C - A}{A}(nt + \gamma)$$

and neglecting c^2,

$$\theta = \omega + \frac{c}{n}\frac{A}{C}\sin\left(\frac{C}{A}nt + \frac{C-A}{A}\gamma\right)$$

$$\psi = \psi_0 - \frac{c}{n\sin\omega}\frac{A}{C}\cos\left(\frac{C}{A}nt + \frac{C-A}{A}\gamma\right)$$

$$\varphi = \varphi_0 + nt - \frac{c\cos\omega}{n\sin\omega}\frac{A}{C}\cos\left(\frac{C}{A}nt + \frac{C-A}{A}\gamma\right)$$

n, c, γ, ω, ψ_0, and φ_0 being constants. In the problem of the Precession of the Equinoxes ω is the mean *obliquity of the ecliptic*, ψ_0 is the *longitude of the first point of Aries* when $t = 0$ reckoned from some fixed line.

$\text{Sin I O } z_1 = \frac{c}{\sqrt{n^2 + c^2}}$, hence it appears that if a body whose form is that of a figure of revolution be made to revolve, and be acted upon by no extraneous force, the axis of instantaneous rotation revolves about the axis of the figure, the *latitude* of the former axis remaining constant.

$$\text{The angle } x_1 \text{O I}_1 = \frac{C-A}{A}(nt + \gamma)$$

If the forces X_1, Y_1, Z_1 arise from the attractions of a distant luminary M', of which the coordinates referred to the axes O x_1, O y_1, O z_1, are x_1', y_1', z_1', the force varying inversely as the square of the distance,

$$X_1 = \frac{M'(x_1' - x_1)}{\{(x_1 - x_1')^2 + (y_1 - y_1')^2 + (z_1 - z_1')^2\}^{\frac{3}{2}}}$$

$$Y_1 = \frac{M'(y_1' - y_1)}{\{(x_1 - x_1')^2 + (y_1 - y_1')^2 + (z_1 - z_1')^2\}^{\frac{3}{2}}}$$

$$Z_1 = \frac{M'(z_1' - z_1)}{\{(x_1 - x_1')^2 + (y_1 - y_1')^2 + (z_1 - z_1')^2\}^{\frac{3}{2}}}$$

$$\int (x_1 Y_1 - y_1 X_1)\,\mathrm{d}m = M'\int \frac{(x_1 y_1' - y_1 x_1')}{r'^3}\left\{1 + \frac{3(x_1 x_1' + y_1 y_1' + z_1 z_1')}{r'^2} + \&c.\right\}\mathrm{d}m$$

By the properties of the principal axes $\int x_1 y_1\,\mathrm{d}m = 0$, $\int x_1 z_1\,\mathrm{d}m = 0$, $\int y_1 z_1\,\mathrm{d}m = 0$, and by the properties of the centre of gravity $\int x_1\,\mathrm{d}m = 0$, $\int y_1\,\mathrm{d}m = 0$, $\int z_1\,\mathrm{d}m = 0$, whence

$$\int (x_1 Y_1 - y_1 X_1)\,\mathrm{d}m = \frac{3M' x_1' y_1'}{r'^5}\int (x_1^2 - y_1^2)\,\mathrm{d}m = \frac{3M'(B-A)}{r'^5} x_1' y_1'\,\mathrm{d}t$$

$$C\,dr + (B-A)\,pq\,dt = \frac{3M'(B-A)}{r'^5}\,x_i'y_i'\,dt$$

$$B\,dq + (A-C)\,rp\,dt = \frac{3M'(A-C)}{r'^5}\,z_i'x_i'\,dt$$

$$A\,dp + (C-B)\,qr\,dt = \frac{3M'(C-B)}{r'^5}\,y_i'z_i'\,dt$$

Substituting for x_i', y_i', z_i' their values from equations, p. 20, upon the supposition of $\psi = 0$,

$$C\,dr + (B-A)\,pq\,dt = \frac{3M'(B-A)}{2r'^5}\left\{\{(y'\cos\theta - x'\sin\theta)^2 - x'^2\}\sin 2\varphi + 2x'(y'\cos\theta - x'\sin\theta)\cos^2\varphi\right\}$$

$$B\,dq + (A-C)\,rp\,dt = \frac{3M'(A-C)}{r'^5}\left\{x'(y'\sin\theta + x'\cos\theta)\cos\varphi + (y'\cos\theta - x'\sin\theta)(y'\sin\theta + x'\cos\theta)\sin\varphi\right\}$$

$$A\,dp + (C-B)\,qr\,dt = \frac{3M'(C-B)}{r'^5}\left\{(y'\cos\theta - x'\sin\theta)(y'\sin\theta + x'\cos\theta)\cos\varphi - x'(y'\sin\theta + x'\cos\theta')\sin\varphi\right\}$$

If

$$\frac{3M'}{r'^5}\left\{(y'\cos\theta - x'\sin\theta)(y'\sin\theta + x'\cos\theta)\right\} = P$$

$$\frac{3M'}{r'^5}\left\{x'(y'\sin\theta + x'\cos\theta)\right\} = P'$$

$$B\,dq + (A-C)\,rp\,dt = (A-C)\,dt\,\{P'\cos\varphi + P\sin\varphi\}$$

$$A\,dp + (C-B)\,qr\,dt = (C-B)\,dt\,\{P\cos\varphi - P'\sin\varphi\}$$

P and P' may be developed according to sines and cosines of angles increasing proportionally to the time. Let $k\cos(it+s)$ be any term of P, $k'\sin(it+s)$ the corresponding term of P',

$$B\,dq + (A-C)\,rp\,dt = \frac{A-C}{2}\,dt\left\{(k+k')\sin(\phi + it + s) + (k-k')\sin(\varphi - it - s)\right\}$$

$$A\,dp + (C-B)\,qr\,dt = \frac{C-B}{2}\,dt\left\{(k+k')\cos(\varphi + it + s) + (k-k')\cos(\varphi - it - s)\right\}$$

The equations which were given p. 21, may still be considered as afford-

ing a solution of the problem by making the constants n, c, γ, ω, ψ_0 and φ_0 vary,

$$dn = 0$$

$$\sin\frac{C-A}{A}(nt+\gamma)\,dc + c\frac{C-A}{A}\cos\frac{C-A}{A}(nt+\gamma)\,d\gamma + \frac{(A-C)}{A}nc\cos\frac{C-A}{A}(nt+\gamma)\,dt$$
$$= \frac{A-C}{2A}dt\left\{(k+k')\sin(\phi+it+s) + (k-k')\sin(\varphi-it-s)\right\}$$

$$\cos\frac{C-A}{A}(nt+\gamma)\,dc - c\frac{C-A}{A}\sin\frac{C-A}{A}(nt+\gamma)\,d\gamma - \frac{A-C}{A}nc\sin\frac{C-A}{A}(nt+\gamma)\,dt$$
$$= -\frac{A-C}{2A}dt\left\{(k+k')\cos(\varphi+it+s) + (k-k')\cos(\varphi-it-s)\right\}$$

since $\varphi = \varphi_0 + nt$ nearly

$$dc = \frac{A-C}{2A}dt\left\{-(k+k')\cos\left(\varphi_0+nt+\frac{C-A}{A}(nt+\gamma)+it+s\right)\right.$$
$$\left.-(k-k')\cos\left(\varphi_0+nt+\frac{C-A}{A}(nt+\gamma)-it-s\right)\right\}$$

$$c\frac{C-A}{A}d\gamma + \frac{A-C}{A}nc\,dt = \frac{A-C}{2A}dt\left\{(k+k')\sin\left(\varphi_0+nt+\frac{C-A}{A}(nt+\gamma)+it+s\right)\right.$$
$$\left.+(k-k')\sin\left(\varphi_0+nt+\frac{C-A}{A}(nt+\gamma)-it-s\right)\right\}$$

$$d\omega + \frac{dc}{n}\frac{A}{C}\sin\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right) + \frac{c}{n}\frac{C-A}{C}\cos\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right)d\gamma = 0$$

$$d\psi_0 - \frac{dc}{n\sin\omega}\frac{A}{C}\cos\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right) + \frac{c\cos\omega}{n\sin\omega^2}\frac{A}{C}\cos\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right)d\omega$$
$$+\frac{c}{n\sin\omega}\frac{C-A}{A}\sin\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right)d\gamma = 0$$

$$d\varphi_0 - \frac{dc}{n\sin\omega}\frac{A}{C}\cos\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right) + \frac{c}{n\sin\omega^2}\frac{A}{C}\cos\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right)d\omega$$
$$+\frac{c}{n\sin\omega}\frac{C-A}{A}\sin\left(\frac{C}{A}nt+\frac{C-A}{A}\gamma\right)d\gamma = 0$$

From the preceding expressions it may be inferred that

n = constant.

$\frac{dc}{dt}$ = series of cosines without any constant quantity, unless the mean mo-

tion of rotation is commensurate to the mean motion of revolution of the luminary M'.

$\frac{d\gamma}{dt}$ = series of sines without any constant quantity, except in a similar case.

c being equal to a constant + a series of sines.

$\frac{d\omega}{dt}$ = a series of sines without any constant quantity.

$\frac{d\psi_0}{dt}$ = a series of cosines + a constant quantity.

$\frac{d\varphi_0}{dt}$ = a series of cosines + a constant quantity.

In the general case where A is not equal to B, n = constant + series of cosines.

The form of the preceding expressions is not affected however, for the approximation may be carried so that except in the case of commensurability above mentioned, the mean motion of rotation being also nearly twice the mean motion of the planet M' in its orbit or greater,

n = constant + series of cosines without any constant quantity multiplied by the time.

c = constant + series of sines or cosines without any constant quantity multiplied by the time.

ω = constant + series of cosines without any constant quantity multiplied by the time.

γ = constant + series of cosines or sines + a constant quantity multiplied by the time.

ψ_0 = constant + series of sines + a constant quantity multiplied by the time.

φ_0 = constant + series of sines + a constant quantity multiplied by the time.

The constant quantity multiplied by the time in the value of ψ_0 is the precession of the equinox.

If $z' = 0$, (which amounts to taking for the fixed plane the orbit of the planet M',) and n' be its mean motion, then neglecting the eccentricity, $x' = a' \cos n' t$, $\quad y' = a' \sin n' t$, $\quad r' = a'$,

$$P = \frac{3M'}{2r'^3} \sin\omega \cos\omega (1 - \cos 2n't), \qquad P' = \frac{3M'}{2r'^3} \sin\omega \sin 2n't$$

Supposing $\varphi_0 = 0$, $\gamma = 0$, $c_0 = 0$, c_0 being in fact imperceptible to observation, and neglecting $\cos 2n't$, $\sin 2n't$, in order to find the constant part of $\frac{d\psi_0}{dt}$,

$$\frac{dc_0}{dt} = -\frac{3(A-C)}{2nAa^3} M' \sin\omega\cos\omega\cos\frac{C}{A}n't$$

$$\frac{c\,d\gamma}{dt} = -\frac{3(A-C)}{2nAa^3} M' \sin\omega\cos\omega\sin\frac{C}{A}n't$$

$$\frac{d\psi_0}{dt} = \frac{3(C-A)}{2nCa^3} M'\cos\omega = \frac{3(C-A)}{2nC} n'^2\cos\omega$$

This result agrees with that given in the Méc. Cél. vol. ii. p. 318, and with that given by M. Poisson, Mémoires de l'Academie, vol. vii. p. 247. In Laplace's notation $\omega = h$, $m = n'$. In M. Poisson's notation $\omega = \theta$, $m = n'$.

On the Theory of the Motion of the Planets, continued from Part II. 1830, *p.* 357.

From the general equations given in the Méc. Cél. vol. i. p. 268, the following may be inferred.

$$\frac{3}{2}\frac{a}{a_i}b_{1,2} = \frac{(a^2+a_i^2)}{a_i^2}b_{1,1} - \frac{a}{a_i}b_{1,0} \qquad \frac{5}{2}\frac{a}{a_i}b_{1,3} = \frac{2(a^2+a_i^2)}{a_i^2}b_{1,2} - \frac{3}{2}\frac{a}{a_i}b_{1,1}$$

$$\frac{7}{2}\frac{a}{a_i}b_{1,4} = \frac{3(a^2+a_i^2)}{a_i^2}b_{1,3} - \frac{5}{2}\frac{a}{a_i}b_{1,2} \qquad \frac{9}{2}\frac{a}{a_i}b_{1,5} = \frac{4(a^2+a_i^2)}{a_i^2}b_{1,4} - \frac{7}{2}\frac{a}{a_i}b_{1,3}$$

$$\frac{a}{2a_i}b_{3,2} = \frac{(a^2+a_i^2)}{a_i^2}b_{3,1} - \frac{3}{2}\frac{a}{a_i}b_{3,0} \qquad \frac{3}{2}\frac{a}{a_i}b_{3,3} = \frac{2(a^2+a_i^2)}{a_i^2}b_{3,2} - \frac{5}{2}\frac{a}{a_i}b_{3,1}$$

$$\frac{5}{2}\frac{a}{a_i}b_{3,4} = \frac{3(a^2+a_i^2)}{a_i^2}b_{3,3} - \frac{7}{2}\frac{a}{a_i}b_{3,2} \qquad \frac{7}{2}\frac{a}{a_i}b_{3,5} = \frac{4(a^2+a_i^2)}{a_i^2}b_{3,4} - \frac{9}{2}\frac{a}{a_i}b_{3,3}$$

$$-\frac{a}{2a_i}b_{5,2} = \frac{(a^2+a_i^2)}{a_i^2}b_{5,1} - \frac{5}{2}\frac{a}{a_i}b_{5,0} \qquad \frac{a}{2a_i}b_{5,3} = \frac{2(a^2+a_i^2)}{a_i^2}b_{5,2} - \frac{7}{2}\frac{a}{a_i}b_{5,1}$$

$$\frac{3}{2}\frac{a}{a_i}b_{5,4} = \frac{3(a^2+a_i^2)}{a_i^2}b_{5,3} - \frac{9}{2}\frac{a}{a_i}b_{5,2} \qquad \frac{5}{2a_i}b_{5,5} = \frac{4(a^2+a_i^2)}{a_i^2}b_{5,4} - \frac{11}{2}\frac{a}{a_i}b_{5,3}$$

$$b_{1,0} = \frac{a^2+a_i^2}{a_i^2}b_{0,0} - \frac{a}{a_i}b_{0,1} \qquad b_{1,1} = \frac{a^2+a_i^2}{a_i^2}b_{0,1} - \frac{2a}{a_i}b_{0,0} - \frac{a}{a_i}b_{0}$$

$$b_{1,2}=\frac{a^2+a_1^2}{a_1^2}b_{2,2}-\frac{a}{a_1}b_{2,1}-\frac{a}{a_1}b_{2,3} \qquad b_{1,3}=\frac{a^2+a_1^2}{a_1^2}b_{2,3}-\frac{a}{a_1}b_{2,2}-\frac{a}{a_1}b_{2,4}$$

$$b_{2,0}=\frac{a^2+a_1^2}{a_1^2}b_{2,0}-\frac{a}{a_1}b_{2,1} \qquad b_{2,1}=\frac{a^2+a_1^2}{a_1^2}b_{2,1}-\frac{2a}{a_1}b_{2,0}-\frac{a}{a_1}b_{2,2}$$

$$b_{2,2}=\frac{a^2+a_1^2}{a_1^2}b_{2,2}-\frac{a}{a_1}b_{2,1}-\frac{a}{a_1}b_{2,3} \qquad b_{2,3}=\frac{a^2+a_1^2}{a_1^2}b_{2,3}-\frac{a}{a_1}b_{2,2}-\frac{a}{a_1}b_{2,4}$$

$$b_{1,1}=\frac{a}{a_1}\{b_{2,0}-\tfrac{1}{2}b_{2,2}\} \qquad 2\,b_{1,2}=\frac{a}{2a_1}\{b_{2,1}-b_{2,3}\}$$

$$3\,b_{1,3}=\frac{a}{2a_1}\{b_{2,2}-b_{2,4}\} \qquad 4\,b_{1,4}=\frac{a}{2a_1}\{b_{2,2}-b_{2,4}\}$$

$$b_{2,1}=3\frac{a}{a_1}\{b_{2,0}-\tfrac{1}{2}b_{2,2}\} \qquad 2\,b_{2,2}=\frac{3}{2}\frac{a}{a_1}\{b_{2,1}-b_{2,3}\}$$

$$3\,b_{2,3}=\frac{3}{2}\frac{a}{a_1}\{b_{2,2}-b_{2,4}\} \qquad 4\,b_{2,4}=\frac{3}{2}\frac{a}{a_1}\{b_{2,3}-b_{2,5}\}$$

$$2\,b_{3,0}=\frac{2\left(1+\frac{a^2}{a_1^2}\right)b_{1,0}-2\frac{a}{a_1}b_{1,1}}{\left(1-\frac{a^2}{a_1^2}\right)^2}, \qquad b_{3,1}=\frac{3\left(1+\frac{a^2}{a_1^2}\right)b_{1,1}-2.3\frac{a}{a_1}b_{1,2}}{\left(1-\frac{a^2}{a_1^2}\right)^2}$$

$$b_{3,2}=\frac{5\left(1+\frac{a^2}{a_1^2}\right)b_{1,2}-2.5\frac{a}{a_1}b_{1,3}}{\left(1-\frac{a^2}{a_1^2}\right)^2}, \qquad b_{3,3}=\frac{7\left(1+\frac{a^2}{a_1^2}\right)b_{1,3}-2.7\frac{a}{a_1}b_{1,4}}{\left(1-\frac{a^2}{a_1^2}\right)^2}$$

$$2\,b_{3,0}=\frac{2\left(1+\frac{a^2}{a_1^2}\right)b_{3,0}+\frac{2}{3}\frac{a}{a_1}b_{3,1}}{\left(1-\frac{a^2}{a_1^2}\right)^2}, \qquad b_{3,1}=\frac{\frac{5}{3}\left(1+\frac{a^2}{a_1^2}\right)b_{3,1}-\frac{2}{3}\frac{a}{a_1}b_{3,2}}{\left(1-\frac{a^2}{a_1^2}\right)^2}$$

$$\tfrac{1}{2}b_{3,2}=\frac{\frac{7}{3}\left(1+\frac{a^2}{a_1^2}\right)b_{3,1}-2\frac{a}{a_1}b_{3,3}}{\left(1-\frac{a^2}{a_1^2}\right)^2}, \qquad b_{3,3}=\frac{\frac{9}{3}\left(1+\frac{a^2}{a_1^2}\right)b_{3,3}-\frac{2.5}{3}\frac{a}{a_1}b_{3,4}}{\left(1-\frac{a^2}{a_1^2}\right)^2}$$

$$2\,b_{7,0}=\frac{2\left(1+\frac{a^2}{a_1^2}\right)b_{5,0}+\frac{2.3}{5}\frac{a}{a_1}b_{5,1}}{\left(1-\frac{a^2}{a_1^2}\right)^2}, \qquad b_{7,1}=\frac{\frac{7}{5}\left(1+\frac{a^2}{a_1^2}\right)b_{5,1}+\frac{2}{5}\frac{a}{a_1}b_{5,2}}{\left(1-\frac{a^2}{a_1^2}\right)^2}$$

$$b_{7,2}=\frac{\frac{9}{5}\left(1+\frac{a^2}{a_i^2}\right)b_{5,2}-\frac{2}{5}\frac{a}{a_i}b_{5,3}}{\left(1-\frac{a^2}{a_i^2}\right)^2},\qquad b_{7,3}=\frac{\frac{11}{5}\left(1+\frac{a^2}{a_i^2}\right)b_{5,3}-\frac{2.3}{5}\frac{a}{a_i}b_{5,4}}{\left(1-\frac{a^2}{a_i^2}\right)^2}$$

$$r=aF(nt);\quad \left(\frac{dR}{dr}\right)=\frac{dRda}{dadaF(nt)},\quad \frac{dadFnt}{da}=Fnt,\quad \frac{rdR}{dr}=a\left(\frac{dR}{da}\right),$$

$$\{a^2-2aa_i\cos\theta+a_i^2\}^{-\frac{1}{2}}=\frac{1}{a_i}\{b_{1,0}+b_{1,1}\cos\theta+b_{1,2}\cos 2\theta+\&c.\}$$

$$\frac{-(a-a_i\cos\theta)}{\{a^2-2aa_i\cos\theta+a_i^2\}^{\frac{3}{2}}}=\frac{1}{a_i}\left\{\frac{d.b_{1,0}}{da}+\frac{db_{1,1}}{da}\cos\theta+\frac{d.b_{1,2}}{da}\cos 2\theta+\&c.\right\}$$

$$\frac{-\{a-a_i\cos\theta\}}{a_i^3}\{b_{3,0}+b_{3,1}\cos\theta+b_{3,2}\cos 2\theta+\&c.\}=\frac{1}{a_i}\left\{\frac{d.b_{1,0}}{da}\right.$$

$$\left.+\frac{d.b_{1,1}}{da}\cos\theta+\frac{d.b_{1,2}}{da}\cos 2\theta+\&c.\right\}$$

whence

$$\frac{ad.b_{1,0}}{da}=-\frac{a}{a_i}\left\{\frac{a}{a_i}b_{3,0}-\frac{1}{2}b_{3,1}\right\}$$

$$\frac{ad.b_{1,1}}{da}=-\frac{a}{a_i}\left\{\frac{a}{a_i}b_{3,1}-b_{3,0}-\frac{1}{2}b_{3,2}\right\}$$

$$\frac{ad.b_{1,2}}{da}=-\frac{a}{a_i}\left\{\frac{a}{a_i}b_{3,2}-\frac{1}{2}b_{3,1}-\frac{1}{2}b_{3,3}\right\}$$

$$\frac{ad.b_{1,3}}{da}=-\frac{a}{a_i}\left\{\frac{a}{a_i}b_{3,3}-\frac{1}{2}b_{3,2}-\frac{1}{2}b_{3,4}\right\}$$

similarly

$$\frac{ad.b_{3,0}}{da}=-3\frac{a}{a_i}\left\{\frac{a}{a}b_{5,0}-\frac{1}{2}b_{5,1}\right\}$$

$$\frac{ad.b_{3,1}}{da}=-3\frac{a}{a_i}\left\{\frac{a}{a_i}b_{5,1}-b_{5,0}-\frac{1}{2}b_{5,2}\right\}$$

$$\frac{ad.b_{5,0}}{da}=-5\frac{a}{a_i}\left\{\frac{a}{a_i}b_{7,0}-\frac{1}{2}b_{7,1}\right\}$$

$$\frac{ad.b_{5,1}}{da}=-5\frac{a}{a_i}\left\{\frac{a}{a_i}b_{7,1}-b_{7,0}-\frac{1}{2}b_{7,2}\right\}$$

$$\frac{a^3 d^3 . b_{1,0}}{da^3} = \frac{a}{a_i}\left\{\frac{2a}{a_i} b_{2,0} - \frac{1}{2} b_{2,1}\right\}$$

$$\frac{a^3 d^3 . b_{1,1}}{da^3} = \frac{a}{a_i}\left\{\frac{2a}{a_i} b_{2,1} - b_{2,2}\right\}$$

$$\frac{a^3 d^3 . b_{1,2}}{da^3} = \frac{a}{a_i}\left\{\frac{2a}{a_i} b_{2,2} + \frac{1}{2} b_{2,1} - \frac{3}{2} b_{2,3}\right\}$$

$$\frac{a^3 d^3 . b_{1,3}}{da^3} = \frac{a}{a_i}\left\{\frac{2a}{a_i} b_{2,3} + b_{2,2} - 2 b_{2,4}\right\}$$

$$\frac{a^3 d^3 . b_{1,4}}{da^3} = \frac{a}{a_i}\left\{\frac{2a}{a_i} b_{2,4} + \frac{3}{2} b_{2,3} - \frac{5}{2} b_{2,5}\right\}$$

The value of R given p. 349 of the former part of this paper is susceptible of much simplification. The first term of R for instance

$$= m_i \left\{ -\frac{b_{1,0}}{a_i} + \frac{3\,(a^2 e^2 + a_i^2 e_i^2)}{2.2\,a_i^3} b_{2,0} + \frac{a\,a_i}{2\,a_i^3}\left(\sin^2\frac{i_i}{2} + \frac{(e^2 + e_i^2)}{2}\right) b_{2,1} \right.$$
$$- \frac{3}{2.4\,a_i^3}\left(2\,a^4 e^2 + 5\,a^2 a_i^2\,(e^2 + e_i^2) + 2\,a_i^4 e_i^2\right) b_{2,0} + \frac{3.2}{2.4\,a_i^3}\,(a^2 e^2 + a_i^2 e_i^2)\,a\,a_i\,b_{2,1}$$
$$\left. + \frac{1.3.3}{2.4.2}\frac{a^2 a_i^2}{a_i^3}\,(e^2 + e_i^2)\,b_{2,2} \right\}$$

$$= m_i \left\{ -\frac{b_{1,0}}{a_i} + \frac{3\,(a^2 e^2 + a_i^2 e_i^2)}{2.2\,a_i^3} b_{2,0} + \frac{a\,a_i}{2\,a_i^3}\left(\sin^2\frac{i_i}{2} + \frac{e^2 + e_i^2}{2}\right) b_{2,1} \right.$$
$$- \frac{3.2}{2.4}\frac{(a^2 + a_i^2)}{a_i^3}\frac{(a^2 e^2 + a_i^2 e_i^2)}{a_i^3} b_{2,0} - \frac{3.3}{2.4}\frac{a^2 a_i^2}{a_i^3}\,(e^2 + e_i^2)\,b_{2,0}$$
$$\left. + \frac{3.2}{2.4}\frac{(a^2 e^2 + a_i^2 e_i^2)}{a_i^3}\frac{a}{a_i} b_{2,1} + \frac{3.3}{2.4.2}\frac{a^2 a_i^2}{a_i^3}\,(e^2 + e_i^2)\,b_{2,2} \right\}$$

and since

$$b_{2,0} = \frac{(a^2 + a_i^2)}{a_i^2} b_{2,0} - \frac{a}{a_i} b_{2,1} \quad \text{See p. 26}$$

$$b_{2,1} = \frac{3a}{a_i}\left\{b_{2,0} - \tfrac{1}{2} b_{2,2}\right\}$$

this term reduces itself to

$$m_i \left\{ -\frac{b_{1,0}}{a_i} + \frac{a\,a_i}{2\,a_i^3}\left(\sin^2\frac{i_i}{2} + \frac{e^2 + e_i^2}{2}\right) b_{2,1} - \frac{3}{2.4}\frac{a\,a_i}{a_i^3}\,(e^2 + e_i^2)\,b_{2,1} \right\}$$
$$= m_i \left\{ -\frac{b_{1,0}}{a_i} + \frac{a}{2\,a_i^2}\left(\sin^2\frac{i_i}{2} - \frac{e^2 + e_i^2}{4}\right) b_{2,1} \right\}$$

The succeeding terms admit of similar simplifications, so that

$$R = m_i \left\{ -\frac{b_{1,0}}{a_i} + \frac{a}{2a_i^2}\left(\sin^2\frac{i_i}{2} + \frac{e^2+e_i^2}{2}\right) b_{3,1} - \frac{3}{2.4}\frac{a}{a_i^2}(e^2+e_i^2) b_{3,1} \right\}$$

$$+ m_i \left\{ \frac{a}{a_i^2}\left(\cos^2\frac{i_i}{2} - \frac{e^2+e_i^2}{2}\right) - \frac{b_{1,1}}{a_i} + \frac{a}{a_i^2}\left(\sin^2\frac{i_i}{2} + \frac{e^2+e_i^2}{2}\right)\left(b_{3,0} + \tfrac{1}{2} b_{3,2}\right) \right.$$

$$\left. - \frac{3.2}{2.4}\frac{a}{a_i^2}(e^2+e_i^2) b_{3,2} \right\} \cos(nt - n_i t + \varepsilon - \varepsilon_i) \qquad [1]$$

$$+ m_i \left\{ -\frac{b_{1,2}}{a_i} + \frac{a}{2a_i^2}\left(\sin^2\frac{i_i}{2} + \frac{e^2+e_i^2}{2}\right)(b_{3,1} + b_{3,3}) + \frac{3}{2.4}\frac{a}{a_i^2}(e^2+e_i^2) b_{3,1} \right.$$

$$\left. - \frac{3.3}{2.4}\frac{a}{a_i^2}(e^2+e_i^2) b_{3,3} \right\} \cos(2nt - 2n_i t + 2\varepsilon - 2\varepsilon_i) \qquad [2]$$

$$+ m_i \left\{ -\frac{b_{1,3}}{a_i} + \frac{a}{2a_i^2}\left(\sin^2\frac{i_i}{2} + \frac{e^2+e_i^2}{2}\right)(b_{3,2} + b_{3,4}) + \frac{3.2}{3.4}\frac{a}{a_i^2}(e^2+e_i^2) b_{3,2} \right.$$

$$\left. - \frac{3.4}{2.4}\frac{a}{a_i^2}(e^2+e_i^2) b_{3,4} \right\} \cos(3nt - 3n_i t + 3\varepsilon - 3\varepsilon_i) \qquad [3]$$

$$+ m_i \left\{ -\frac{b_{1,4}}{a_i} + \frac{a}{2a_i^2}\left(\sin^2\frac{i_i}{2} + \frac{e^2+e_i^2}{2}\right)(b_{3,3} + b_{3,5}) + \frac{3.3}{2.4}\frac{a}{a_i^2}(e^2+e_i^2) b_{3,3} \right.$$

$$\left. - \frac{3.5}{2.4}\frac{a}{a_i^2}(e^2+e_i^2) b_{3,5} \right\} \cos(4nt - 4n_i t + 4\varepsilon - 4\varepsilon_i) \qquad [4]$$

The coefficient of $\cos(\varpi - \varpi_i)$, Argument 41,

$$= m_i \left\{ -\frac{9}{4}\frac{a}{a_i^3} b_{3,0} - \frac{a}{8a_i^3} b_{3,2} + \frac{3.6}{2.4}\frac{(a^2+a_i^2)}{a_i^3} a a_i b_{3,0} - \frac{3.7}{2.4.2}\frac{a^2 a_i^2}{a_i^5} b_{3,1} \right.$$

$$\left. - \frac{3}{2.4}\frac{(a^2+a_i^2)}{a_i^3} a a_i b_{3,2} - \frac{3}{2.4.2}\frac{a^2 a_i^2}{a_i^5} b_{3,3} \right\} e e_i$$

$$= m_i \left\{ -\frac{9}{4}\frac{a}{a_i^3} b_{3,0} - \frac{a}{8a_i^3} b_{3,2} + \frac{3.6}{2.4}\frac{a}{a_i^2}\left\{ \frac{(a^2+a_i^2)}{a_i^2} b_{3,0} - \frac{a}{a_i} b_{3,1} \right\} \right.$$

$$\left. - \frac{3}{8}\frac{a}{a_i^3}\left\{ \frac{(a^2+a_i^2)}{a_i^2} b_{3,2} - \frac{a}{a_i} b_{3,1} - \frac{a}{a_i} b_{3,3} \right\} + \frac{9}{16}\frac{a}{a_i^3}\left\{ b_{3,1} - b_{3,3} \right\} \right\}$$

$$= m_i \left\{ -\frac{9}{4}\frac{a}{a_i^3} b_{3,0} - \frac{a}{8a_i^3} b_{3,2} + \frac{9}{4}\frac{a}{a_i^3} b_{3,0} - \frac{3}{8}\frac{a}{a_i^3} b_{3,2} + \frac{2.3}{8}\frac{a}{a_i^3} b_{3,2} \right\} e e_i$$

$$= m_i \frac{a}{4a_i^3} b_{3,2}$$

So that the part of R which is independent of $n\,t$, $n_i\,t$

$$= m_i \left\{ -\frac{b_{1,0}}{a_i^2} + \frac{a}{2a_i^2}\left(\sin^2\frac{i_i}{2} - \frac{e^2+e_i^2}{4}\right) b_{3,1} + \frac{a}{4a_i^2} b_{3,2} e e_i \cos(\varpi - \varpi_i) \right\}$$

$$= m_i \left\{ -\frac{b_{1,0}}{a_i} - \frac{a}{2a_i^2}\left(\sin^2\frac{i_i}{2} - \frac{e^2+e_i^2}{4}\right) b_{3,1} - \left\{ \frac{3a}{2a_i^2} b_{3,0} - \frac{a^2+a_i^2}{2a_i^3} b_{3,1} \right\} e e_i \cos(\varpi - \varpi_i) \right\}$$

In the general case, when i_1, i_2 are the inclinations of the orbits of the planets P and P_i to any plane, the direction of which is arbitrary,

$$\cos i_i = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos (\nu_1 - \nu_2)$$

$$\sin^2 \frac{i_i}{2} = \frac{1 - \cos i_1 \cos i_2 - \sin i_1 \sin i_2 \cos (\nu_1 - \nu_2)}{2}$$

The part of R which is independent of nt, $n_i t$

$$= m_i \left\{ -\frac{b_{1,0}}{a_i} + \frac{a}{4 a_i^2} \left\{ 1 - \cos i_1 \cos i_2 - \sin i_1 \sin i_2 \cos (\nu_1 - \nu_2) + \frac{e^2 + e_i^2}{2} \right\} b_{3,1} \right.$$
$$\left. - \left\{ \frac{3a}{2 a_i^2} b_{3,0} - \frac{(a^2 + a_i^2)}{2 a_i^3} b_{3,1} \right\} e e_i \cos (\varpi - \varpi_i) \right\}$$

$$\cos i = \frac{1}{\sqrt{1 + \tan^2 i}} = 1 - \tfrac{1}{2} \tan^2 i, \qquad \sin i = \frac{\tan i}{\sqrt{1 + \tan^2 i}} = \tan i \quad \text{nearly.}$$

$$= m_i \left\{ -\frac{b_{1,0}}{a_i} + \frac{a}{8 a_i^2} \left\{ (\tan^2 i_1 + \tan^2 i_2 - 2 \tan i_1 \tan i_2 \cos (\nu_1 - \nu_2) - e^2 - e_i^2 \right\} b_{3,1} \right.$$
$$\left. - \left\{ \frac{3a}{2 a_i^2} b_{3,0} - \frac{(a^2 + a_i^2)}{2 a_i^3} b_{3,1} \right\} e e_i \cos (\varpi - \varpi_i) \right\}$$

$$= m_i \left\{ -\frac{b_{1,0}}{a_i} + \frac{a}{8 a_i^2} \left\{ (\tan i_1 \cos \nu_1 - \tan i_2 \cos \nu_2)^2 + (\tan i_1 \sin \nu_1 - \tan i_2 \sin i_2)^2 - e^2 - e_i^2 \right\} b_{3,1} \right.$$
$$\left. - \left\{ \frac{3a}{2 a_i^2} b_{3,0} - \frac{(a^2 + a_i^2)}{2 a_i^3} b_{3,1} \right\} e e_i \cos (\varpi - \varpi_i) \right\}$$

and if $\tan i \sin \nu = p$, $\tan i \cos \nu = q$, this quantity

$$= m_i \left\{ -\frac{b_{1,0}}{a_i} + \frac{a}{8 a_i^2} \left\{ (p_1 - p_2)^2 + (q_1 - q_2)^2 - e^2 - e_i^2 \right\} b_{3,1} \right.$$
$$\left. - \left\{ \frac{3a}{2 a_i^2} b_{3,0} - \frac{a^2 + a_i^2}{2 a_i^3} b_{3,1} \right\} e e_i \cos (\varpi - \varpi_i) \right\}$$

which evidently agrees with the result given by M. de PONTÉCOULANT, Théor. Anal. vol. i. p. 363. All the other coefficients of terms multiplied by the squares and products of the eccentricities are susceptible of reductions similar to those in the two preceding pages, and finally;

$$R = m_i \left\{ -\frac{b_{1,0}}{a_i} + \frac{a}{2 a_i^2} \left(\sin^2 \frac{i_i}{2} - \frac{e^2 + e_i^2}{4} \right) b_{3,1} \right\} \qquad [0]$$

$$+ m_i \left\{ -\frac{a}{a_i^2} \left(\cos^2 \frac{i_i}{2} - \frac{e^2 + e_i^2}{2} \right) - \frac{b_{3,1}}{a_i} + \frac{a}{a_i^2} \sin^2 \frac{i_i}{2} (b_{3,0} + \tfrac{1}{2} b_{3,2}) \right.$$
$$\left. + \frac{a}{a_i^2} \frac{(e^2 + e_i^2)}{8} (4 b_{3,0} - 4 b_{3,2}) \right\} \cos (n t - n_i t + \varepsilon - \varepsilon_i) \qquad [1]$$

Development of R.

$$+ m_i \left\{ -\frac{b_{1,2}}{a_i} + \frac{a}{2a_i^2}\sin^2\frac{i_i}{2}(b_{2,1}+b_{2,3}) + \frac{a}{a_i^2}\frac{(e^2+e_i^2)}{8}(5b_{2,1}-7b_{2,3}) \right\} \cos(2nt-2n_it+2\varepsilon-2\varepsilon_i) \quad [2]$$

$$+ m_i \left\{ -\frac{b_{1,3}}{a_i} + \frac{a}{2a_i^2}\sin^2\frac{i_i}{2}(b_{3,2}+b_{3,4}) + \frac{a}{a_i^2}\frac{(e^2+e_i^2)}{8}(8b_{3,2}-10b_{3,4}) \right\} \cos(3nt-3n_it+3\varepsilon-3\varepsilon_i) \quad [3]$$

$$+ m_i \left\{ -\frac{b_{1,4}}{a_i} + \frac{a}{2a_i^2}\sin^2\frac{i_i}{2}(b_{4,3}+b_{4,5}) + \frac{a}{a_i^2}\frac{(e^2+e_i^2)}{8}(11b_{4,3}-13b_{4,5}) \right\} \cos(4nt-4n_it+4\varepsilon-4\varepsilon_i) \quad [4]$$

$$+ m_i \left\{ -\frac{b_{1,5}}{a_i} + \frac{a}{2a_i^2}\sin^2\frac{i_i}{2}(b_{5,4}+b_{5,6}) + \frac{a}{a_i^2}\frac{(e^2+e_i^2)}{8}(14b_{5,4}-16b_{5,6}) \right\} \cos(5nt-5n_it+5\varepsilon-5\varepsilon_i) \quad [5]$$

$$+ m_i \left\{ -\frac{3a}{2a_i^2} + \frac{3a}{2a_i^2}b_{3,0} - \frac{a^2}{2a_i^3}b_{3,1} - \frac{a}{4a_i^2}b_{3,2} \right\} e\cos(n_it+\varepsilon-\varpi) \quad [6]\ [16]^*$$

$$+ m_i \left\{ -\frac{a^2}{a_i^3}b_{3,0} + \frac{a}{2a_i^2}b_{3,1} \right\} e\cos(nt+\varepsilon-\varpi) \quad [7]\ [15]$$

$$+ m_i \left\{ \frac{a}{2a_i^2} - \frac{a}{2a_i^2}b_{3,0} - \frac{a^2}{2a_i^3}b_{3,1} + \frac{3}{4}\frac{a}{a_i^2}b_{3,2} \right\} e\cos(2nt-n_it+2\varepsilon-\varepsilon_i-\varpi) \quad [8]\ [20]$$

$$+ m_i \left\{ -\frac{a}{4a_i^2}b_{3,1} - \frac{a^2}{2a_i^3}b_{3,2} + \frac{3}{4}\frac{a}{a_i^2}b_{3,3} \right\} e\cos(3nt-2n_it+3\varepsilon-2\varepsilon_i-\varpi) \quad [9]\ [21]$$

$$+ m_i \left\{ -\frac{a}{4a_i^2}b_{3,2} - \frac{a^2}{2a_i^3}b_{3,3} + \frac{3}{4}\frac{a}{a_i^2}b_{3,4} \right\} e\cos(4nt-3n_it+4\varepsilon-3\varepsilon_i-\varpi) \quad [10]\ [22]$$

$$+ m_i \left\{ -\frac{a}{4a_i^2}b_{3,3} - \frac{a}{2a_i^3}b_{3,4} + \frac{3}{4}\frac{a}{a_i^2}b_{3,5} \right\} e\cos(5nt-4n_it+5\varepsilon-4\varepsilon_i-\varpi) \quad [11]\ [23]$$

$$+ m_i \left\{ \frac{3a}{4a_i^2}b_{3,1} - \frac{a^2}{2a_i^3}b_{3,2} - \frac{a}{4a_i^2}b_{3,3} \right\} e\cos(nt-2n_it+\varepsilon-2\varepsilon_i+\varpi) \quad [12]\ [17]$$

$$+ m_i \left\{ \frac{3a}{4a_i^2}b_{3,2} - \frac{a^2}{2a_i^3}b_{3,3} - \frac{a}{4a_i^2}b_{3,4} \right\} e\cos(2nt-3n_it+2\varepsilon-3\varepsilon_i+\varpi) \quad [13]\ [18]$$

$$+ m_i \left\{ \frac{3a}{4a_i^2}b_{3,3} - \frac{a^2}{2a_i^3}b_{3,4} - \frac{a}{4a_i^2}b_{3,5} \right\} e_i\cos(3nt-4n_it+3\varepsilon-4\varepsilon_i+\varpi) \quad [14]\ [19]$$

$$+ m_i \left\{ -\frac{a_i^2}{a_i^3}b_{3,0} + \frac{a}{2a_i^2}b_{3,1} \right\} e_i\cos(n_it+\varepsilon_i-\varpi_i) \quad [15]\ [7]$$

* These numbers indicate the arguments which are symmetrical with regard to nt and n_it.

Development of R.

$$+ m_i \left\{ \frac{3a}{2a_i^2} b_{3,0} - \frac{a_i^2}{2a_i^3} b_{3,1} - \frac{a}{4a_i^2} b_{3,2} \right\} e_i \cos(nt + \varepsilon - \varpi_i) \quad [16]\ [6]$$

$$+ m_i \left\{ \frac{3a}{4a_i^2} b_{3,1} - \frac{a_i^2}{2a_i^3} b_{3,2} - \frac{a}{4a_i^2} b_{3,3} \right\} e_i \cos(2nt - n_i t + 2\varepsilon - \varepsilon_i - \varpi_i) \quad [17]\ [12]$$

$$+ m_i \left\{ \frac{3a}{4a_i^2} b_{3,2} - \frac{a_i^2}{2a_i^3} b_{3,3} - \frac{a}{4a_i^2} b_{3,4} \right\} e_i \cos(3nt - 2n_i t + 3\varepsilon - 2\varepsilon_i - \varpi_i) \quad [18]\ [13]$$

$$+ m_i \left\{ \frac{3a}{4a_i^2} b_{3,3} - \frac{a_i^2}{2a_i^3} b_{3,4} - \frac{a}{4a_i^2} b_{3,5} \right\} e_i \cos(4nt - 3n_i t + 4\varepsilon - 3\varepsilon_i - \varpi_i) \quad [19]\ [14]$$

$$+ m_i \left\{ \frac{a}{2a_i^2} - \frac{a}{2a_i^3} b_{3,0} - \frac{a_i^2}{2a^3} b_{3,1} + \frac{3a}{4a_i^2} b_{3,2} \right\} e_i \cos(nt - 2n_i t + \varepsilon - 2\varepsilon_i + \varpi_i) \quad [20]\ [8]$$

$$+ m_i \left\{ -\frac{a}{4a_i^2} b_{3,1} - \frac{a_i^2}{2a_i^3} b_{3,2} + \frac{3a}{4a_i^2} b_{3,3} \right\} e_i \cos(2nt - 3n_i t + 2\varepsilon - 3\varepsilon_i + \varpi_i) \quad [21]\ [9]$$

$$+ m_i \left\{ -\frac{a}{4a_i^2} b_{3,2} - \frac{a_i^2}{2a_i^3} b_{3,3} + \frac{3a}{4a_i^2} b_{3,4} \right\} e_i \cos(3nt - 4n_i t + 3\varepsilon - 4\varepsilon_i + \varpi_i) \quad [22]\ [10]$$

$$+ m_i \left\{ -\frac{a}{4a_i^2} b_{3,3} - \frac{a_i^2}{2a_i^3} b_{3,4} + \frac{3a}{4a_i^2} b_{3,5} \right\} e_i \cos(4nt - 5n_i t + 4\varepsilon - 5\varepsilon_i + \varpi_i) \quad [23]\ [11]$$

$$+ m_i \left\{ -\frac{5}{8}\frac{a}{a_i^2} b_{3,1} + \frac{a^2}{2a_i^3} b_{3,2} \right\} e^2 \cos(2n_i t + 2\varepsilon_i - 2\varpi) \quad [24]\ [59]$$

$$+ m_i \left\{ \frac{a}{8a_i^2} - \frac{a}{8a_i^2} b_{3,0} - \frac{a}{16a_i^2} b_{3,2} \right\} e^2 \cos(nt + n_i t + \varepsilon + \varepsilon_i - 2\varpi) \quad [25]\ [58]$$

$$+ m_i \left\{ -\frac{a^2}{a_i^3} b_{3,0} + \frac{3}{8}\frac{a}{a_i^2} b_{3,1} \right\} e^2 \cos(2nt + 2\varepsilon - 2\varpi) \quad [26]\ [57]$$

$$+ m_i \left\{ \frac{3}{8}\frac{a}{a_i^2} - \frac{3}{8}\frac{a}{a_i^2} b_{3,0} - \frac{a^2}{a_i^3} b_{3,1} + \frac{17}{16}\frac{a}{a_i^2} b_{3,2} \right\} e^2 \cos(3nt - n_i t + 3\varepsilon - \varepsilon_i - 2\varpi) \quad [27]\ [63]$$

$$+ m_i \left\{ -\frac{a}{4a_i^2} b_{3,1} - \frac{3}{2}\frac{a^2}{a_i^3} b_{3,2} + \frac{13}{8}\frac{a}{a_i^2} b_{3,3} \right\} e^2 \cos(4nt - 2n_i t + 4\varepsilon - 2\varepsilon_i - 2\varpi) \quad [28]\ [64]$$

$$+ m_i \left\{ -\frac{5}{16}\frac{a}{a_i^2} b_{3,2} - 2\frac{a^2}{a_i^3} b_{3,3} + \frac{35}{16}\frac{a}{a_i^2} b_{3,4} \right\} e^2 \cos(5nt - 3n_i t + 5\varepsilon - 3\varepsilon_i - 2\varpi) \quad [29]\ [65]$$

$$+ m_i \left\{ -\frac{3}{8}\frac{a}{a_i^2} b_{3,3} - \frac{5}{2}\frac{a^2}{a_i^3} b_{3,4} + \frac{11}{4}\frac{a}{a_i^2} b_{3,5} \right\} e^2 \cos(6nt - 4n_i t + 6\varepsilon - 4\varepsilon_i - 2\varpi) \quad [30]\ [66]$$

$$+ m_i \left\{ -\frac{7}{16}\frac{a}{a_i^2} b_{3,4} - 3\frac{a^2}{a_i^3} b_{3,5} + \frac{53}{16}\frac{a}{a_i^2} b_{3,6} \right\} e^2 \cos(7nt - 5n_i t + 7\varepsilon - 5\varepsilon_i - 2\varpi) \quad [31]\ [67]$$

$$+ m_i \left\{ -\frac{19}{16}\frac{a}{a_i^2} b_{3,3} + \frac{a^2}{a_i^3} b_{3,3} + \frac{a}{16a_i^2} b_{3,4} \right\} e^2 \cos(nt - 3n_i t + \varepsilon - 3\varepsilon_i + 2\varpi) \quad [32]\ [60]$$

$$+ m_i \left\{ -\frac{7}{4}\frac{a}{a_i^2} b_{3,3} + \frac{3}{2}\frac{a^2}{a_i^3} b_{3,4} + \frac{1}{8}\frac{a}{a_i^2} b_{3,5} \right\} e^2 \cos(2nt - 4n_i t + 2\varepsilon - 4\varepsilon_i + 2\varpi) \quad [33]\ [61]$$

$$+ m_i \left\{ -\frac{37}{16}\frac{a}{a_i^{\,2}} b_{3,4} + 2\,\frac{a^2}{a_i^{\,3}} b_{3,3} + \frac{3}{16}\frac{a}{a_i^{\,2}} b_{3,6} \right\} e^3 \cos(3nt - 5n_it + 3\varepsilon - 5\varepsilon_i + 2\varpi) \quad [34]\ [62]$$

Development of R.

$$+ m_i \frac{a}{4a_i^{\,2}} b_{3,1}\, e e_i \cos(nt - n_it + \varepsilon - \varepsilon_i - \varpi + \varpi_i) \quad [35]$$

$$+ m_i \left\{ \frac{a}{a_i^{\,2}} - \frac{a}{a_i^{\,2}} b_{3,0} + \frac{3}{4}\frac{a}{a_i^{\,2}} b_{3,2} \right\} e e_i \cos(2nt - 2n_it + 2\varepsilon - 2\varepsilon_i - \varpi + \varpi_i) \quad [36]$$

$$- m_i \left\{ \frac{7}{8}\frac{a}{a_i^{\,2}} b_{3,1} - \frac{9}{8}\frac{a}{a_i^{\,2}} b_{3,3} \right\} e e_i \cos(3nt - 3n_it + 3\varepsilon - 3\varepsilon_i - \varpi + \varpi_i) \quad [37]$$

$$- m_i \left\{ \frac{5}{4}\frac{a}{a_i^{\,2}} b_{3,2} - \frac{3}{2}\frac{a}{a_i^{\,2}} b_{3,4} \right\} e e_i \cos(4nt - 4n_it + 4\varepsilon - 4\varepsilon_i - \varpi + \varpi_i) \quad [38]$$

$$- m_i \left\{ \frac{13}{8}\frac{a}{a_i^{\,2}} b_{3,3} - \frac{15}{8}\frac{a}{a_i^{\,2}} b_{3,5} \right\} e e_i \cos(5nt - 5n_it + 5\varepsilon - 5\varepsilon_i - \varpi + \varpi_i) \quad [39]$$

$$- m_i \left\{ 2\,\frac{a}{a_i^{\,2}} b_{3,4} - \frac{9}{4}\frac{a}{a_i^{\,2}} b_{3,6} \right\} e e_i \cos(6nt - 6n_it + 6\varepsilon - 6\varepsilon_i - \varpi - \varpi_i) \quad [40]$$

$$+ m_i \frac{a}{4a_i^{\,2}} b_{3,2}\, e e_i \cos(\varpi - \varpi_i) \quad [41]$$

$$- m_i \left\{ \frac{3}{8}\frac{a}{a_i^{\,2}} b_{3,1} - \frac{5}{8}\frac{a}{a_i^{\,2}} b_{3,3} \right\} e e_i \cos(nt - n_it + \varepsilon - \varepsilon_i + \varpi - \varpi_i) \quad [42]$$

$$- m_i \left\{ \frac{3}{4}\frac{a}{a_i^{\,2}} b_{3,2} - \frac{a}{a_i^{\,2}} b_{3,4} \right\} e e_i \cos(2nt - 2n_it + 2\varepsilon - 2\varepsilon_i + \varpi - \varpi_i) \quad [43]$$

$$- m_i \left\{ \frac{9}{8}\frac{a}{a_i^{\,2}} b_{3,3} - \frac{11}{8}\frac{a}{a_i^{\,2}} b_{3,5} \right\} e e_i \cos(3nt - 3n_it + 3\varepsilon - 3\varepsilon_i + \varpi - \varpi_i) \quad [44]$$

$$- m_i \left\{ \frac{3}{2}\frac{a}{a_i^{\,2}} b_{3,4} - \frac{7}{4}\frac{a}{a_i^{\,2}} b_{3,6} \right\} e e_i \cos(4nt - 4n_it + 4\varepsilon - 4\varepsilon_i + \varpi - \varpi_i) \quad [45]$$

$$+ m' \left\{ -\frac{3a}{a_i^{\,2}} + \frac{1}{a_i} b_{3,1} - \frac{3}{4}\frac{a}{a_i^{\,2}} b_{3,2} \right\} e e_i \cos(2n_it + 2\varepsilon_i - \varpi - \varpi_i) \quad [46]\ [48]$$

$$+ m_i \frac{a}{4a_i^{\,2}} b_{3,1}\, e e_i \cos(nt + n_it + \varepsilon + \varepsilon_i - \varpi - \varpi_i) \quad [47]$$

$$+ m_i \left\{ \frac{a^2}{a_i^{\,3}} b_{3,1} - \frac{3}{4}\frac{a}{a_i^{\,2}} b_{3,2} \right\} e e_i \cos(2nt + 2\varepsilon - \varpi - \varpi_i) \quad [48]\ [46]$$

$$+ m_i \left\{ \frac{21}{8}\frac{a}{a_i^{\,2}} b_{3,1} - \frac{2}{a_i} b_{3,2} - \frac{3}{8}\frac{a}{a_i^{\,2}} b_{3,3} \right\} e e_i \cos(3nt - n_it + 3\varepsilon - \varepsilon_i - \varpi - \varpi) \quad [49]\ [53]$$

$$+ m_i \left\{ \frac{15}{4}\frac{a}{a_i^{\,2}} b_{3,2} - \frac{3}{a_i} b_{3,3} - \frac{1}{2}\frac{a}{a_i^{\,2}} b_{3,4} \right\} e e_i \cos(4nt - 2n_it + 4\varepsilon - 2\varepsilon_i - \varpi - \varpi) \quad [50]\ [54]$$

Development of R.

$$+m_i\left\{\frac{39}{8}\frac{a}{a_i^2}b_{3,3}-\frac{4}{a_i}b_{3,4}-\frac{5}{8}\frac{a}{a_i^2}b_{3,5}\right\}ee_i\cos(5nt-3n_it+5\varepsilon-3\varepsilon_i-\varpi-\varpi_i)\quad[51]\ [55]$$

$$+m_i\left\{6\frac{a}{a_i^2}b_{3,4}-\frac{5}{a_i}b_{3,5}-\frac{3}{4}\frac{a}{a_i^2}b_{3,6}\right\}ee_i\cos(6nt-4n_it+6\varepsilon-4\varepsilon_i-\varpi-\varpi_i)\quad[52]\ [56]$$

$$+m_i\left\{\frac{21}{8}\frac{a}{a_i^2}b_{3,1}-\frac{2a^2}{a_i^3}b_{3,2}-\frac{3a}{8a_i^2}b_{3,3}\right\}ee_i\cos(nt-3n_it+\varepsilon-3\varepsilon_i+\varpi+\varpi_i)\quad[53]\ [49]$$

$$+m_i\left\{\frac{15}{4}\frac{a}{a_i^2}b_{3,2}-\frac{3a^2}{a_i^3}b_{3,3}-\frac{a}{2a_i^2}b_{3,4}\right\}ee_i\cos(2nt-4n_it+2\varepsilon-4\varepsilon_i+\varpi+\varpi_i)\quad[54]\ [50]$$

$$+m_i\left\{\frac{39}{8}\frac{a}{a_i^2}b_{3,2}-\frac{4a^2}{a_i^3}b_{3,4}-\frac{5}{8}\frac{a}{a_i^2}b_{3,5}\right\}ee_i\cos(3nt-5n_it+3\varepsilon-5\varepsilon_i-\varpi+\varpi_i)\quad[55]\ [51]$$

$$+m_i\left\{6\frac{a}{a_i^2}b_{3,4}-\frac{5a^2}{a_i^3}b_{3,5}-\frac{3}{4}\frac{a}{a_i^2}b_{3,6}\right\}ee_i\cos(4nt-6n_it+4\varepsilon-6\varepsilon_i-\varpi+\varpi_i)\quad[56]\ [52]$$

$$+m_i\left\{-\frac{1}{a_i}b_{3,0}+\frac{3}{8}\frac{a}{a_i^2}b_{3,1}\right\}e_i^2\cos(2n_it+2\varepsilon_i-2\varpi_i)\quad[57]\ [26]$$

$$+m_i\left\{\frac{a}{8a_i^2}-\frac{a}{8a_i^2}b_{3,0}-\frac{a}{16a_i^2}b_{3,2}\right\}e_i^2\cos(nt+n_it+\varepsilon+\varepsilon_i-2\varpi_i)\quad[58]\ [25]$$

$$+m_i\left\{-\frac{5}{8}\frac{a}{a_i^2}b_{3,1}+\frac{1}{2a_i}b_{3,2}\right\}e_i^2\cos(2nt+2\varepsilon-2\varpi_i)\quad[59]\ [24]$$

$$+m_i\left\{-\frac{19}{16}\frac{a}{a_i^2}b_{3,2}+\frac{1}{a_i}b_{3,3}+\frac{a}{16a_i^2}b_{3,4}\right\}e_i^2\cos(3nt-n_it+3\varepsilon-\varepsilon_i-2\varpi_i)\quad[60]\ [32]$$

$$+m_i\left\{-\frac{7}{4}\frac{a}{a_i^2}b_{3,3}+\frac{3}{2a_i}b_{3,4}+\frac{a}{8a_i^2}b_{3,5}\right\}e_i\cos(4nt-2n_it+4\varepsilon-2\varepsilon_i-2\varpi_i)\quad[61]\ [33]$$

$$+m_i\left\{-\frac{37}{16}\frac{a}{a_i^2}b_{3,4}+\frac{2}{a_i}b_{3,5}+\frac{3}{16}\frac{a}{a_i^2}b_{3,6}\right\}e_i^2\cos 5nt-3n_it+5\varepsilon-3\varepsilon_i-2\varpi_i)\quad[62]\ [34]$$

$$+m_i\left\{\frac{27}{8}\frac{a}{a_i^2}-\frac{3}{8}\frac{a}{a_i^2}b_{3,0}-\frac{1}{a_i}b_{3,1}+\frac{17}{16}\frac{a}{a_i^2}b_{3,2}\right\}e_i^2\cos(nt-3n_it+\varepsilon-3\varepsilon_i+2\varpi_i)\quad[63]\ [27]$$

$$+m_i\left\{-\frac{a}{4a_i^2}b_{3,1}-\frac{3}{2a_i}b_{3,2}+\frac{13}{8}\frac{a}{a_i^2}b_{3,3}\right\}e_i^2\cos(2nt-4n_it+2\varepsilon-4\varepsilon_i+2\varpi_i)\quad[64]\ [28]$$

$$+m_i\left\{-\frac{5}{16}\frac{a}{a_i^2}b_{3,2}-\frac{2}{a_i}b_{3,3}+\frac{35}{16}\frac{a}{a_i^2}b_{3,4}\right\}e_i^2\cos(3nt-5n_it+3\varepsilon-5\varepsilon_i+2\varpi_i)\quad[65]\ [29]$$

$$+m_i\left\{-\frac{3}{8}\frac{a}{a_i^2}b_{3,3}-\frac{5}{2a_i}b_{3,4}+\frac{14}{4}\frac{a}{a_i^2}b_{3,5}\right\}e_i^2\cos(4nt-6n_it+4\varepsilon-6\varepsilon_i+2\varpi)\quad[66]\ [30]$$

$$+ m_i \left\{ -\frac{7}{16}\frac{a}{a_i^3} b_{3,4} - \frac{3}{a_i} b_{3,5} + \frac{53}{16}\frac{a}{a_i^3} b_{3,6} \right\} e_i^3 \cos(5nt - 7n_i t + 5\varepsilon - 7\varepsilon_i + 2\varpi_i) \quad [67] \quad [31]$$

Development of R.

$$+ m_i \left\{ \frac{a}{a_i^2} - \frac{a}{a_i^3} b_{3,0} \right\} \sin^2 \frac{i_i}{2} (nt + n_i t + \varepsilon + \varepsilon_i - 2\nu_i) \quad [68]$$

$$- \frac{m_i}{2}\frac{a}{a_i^3} b_{3,1} \sin^2 \frac{i_i}{2} \cos(2n_i t + 2\varepsilon_i - 2\nu_i) \quad [69]$$

$$- \frac{m_i}{2}\frac{a}{a_i^3} b_{3,1} \sin^2 \frac{i_i}{2} \cos(2nt + 2\varepsilon - 2\nu_i) \quad [70]$$

$$- \frac{m_i}{2}\frac{a}{a_i^3} b_{3,2} \sin^2 \frac{i_i}{2} \cos(nt + 3n_i t + \varepsilon - 3\varepsilon_i + 2\nu_i) \quad [71]$$

$$- \frac{m_i}{2}\frac{a}{a_i^3} b_{3,2} \sin^2 \frac{i_i}{2} \cos(3nt - n_i t + 3\varepsilon - \varepsilon_i - 2\nu_i) \quad [72]$$

$$- \frac{m_i}{2}\frac{a}{a_i^3} b_{3,3} \sin^2 \frac{i_i}{2} \cos(2nt - 4n_i t + 2\varepsilon - 4\varepsilon_i + 2\nu_i) \quad [73]$$

$$- \frac{m_i}{2}\frac{a}{a_i^3} b_{3,3} \sin^2 \frac{i_i}{2} \cos(4nt - 2n_i t + 4\varepsilon - 2\varepsilon_i - 2\nu_i) \quad [74]$$

In the lunar theory, the small value of the quantity $\frac{a}{a_i}$ makes it desirable to ordain the results according to powers of this quantity. Transforming therefore the preceding expression for R by means of the equations given in the former part of this paper, Phil. Trans. for 1830, p. 346, neglecting terms multiplied by $\frac{a^4}{a_i^4}$, and supposing $\iota_i = 0$,

$$R = m_i \left\{ -\frac{1}{a_i}\left(1 + \frac{a^2}{4a_i^2}\right) + \frac{3}{2}\left(\sin^2\frac{i}{2} - \frac{(e^2 + e_i^2)}{4}\right)\frac{a^2}{a_i^3} \right\} \quad [0]$$

$$+ m_i \left\{ \left(\cos^2\frac{i}{2} - \frac{(e^2 + e_i^2)}{2}\right)\frac{a}{a_i^2} - \frac{a}{a_i^2}\left(1 + \frac{3}{8}\frac{a}{a_i^2}\right) + \sin^2\frac{i}{2}\frac{a}{a_i^2}\left(1 + \frac{33}{8}\frac{a^2}{a_i^2}\right) \right.$$
$$\left. + \frac{(e^2 + e_i^2)}{2}\frac{a}{a_i^3}\left(1 - \frac{3}{2}\frac{a^2}{a_i^2}\right) \right\} \cos(nt - n_i t + \varepsilon - \varepsilon_i) \quad [1]$$

$$+ m_i \left\{ -\frac{3}{4}\frac{a^2}{a_i^3} + \frac{3}{2}\sin^2\frac{i}{2}\frac{a^2}{a_i^3} + \frac{15}{8}(e^2 + e_i^2)\frac{a^2}{a_i^3} \right\} \cos(2nt - 2n_i t + 2\varepsilon - 2\varepsilon_i) \quad [2]$$

$$+ m_i \left\{ -\frac{5}{8}\frac{a^3}{a_i^4} + \frac{15}{8}\sin^2\frac{i}{2}\frac{a^3}{a_i^4} + \frac{15}{4}(e^2 + e_i^2)\frac{a^3}{a_i^4} \right\} \cos(3nt - 3n_i t + 3\varepsilon - 3\varepsilon_i) \quad [3]$$

Development of R according to powers of $\frac{a}{a_i}$.

$$+ m_i \frac{15}{16} \frac{a^3}{a_i^4} e \cos (n_i t + \varepsilon_i - \varpi) \quad [6]$$

$$+ \frac{m_i}{2} \frac{a^2}{a_i^3} e \cos (n t + \varepsilon - \varpi) \quad [7]$$

$$+ m_i \frac{3}{16} \frac{a^3}{a_i^4} e \cos (2 n t - n_i t + 2 \varepsilon - \varepsilon_i - \varpi) \quad [8]$$

$$- m_i \frac{3}{4} \frac{a^2}{a_i^3} e \cos (3 n t - 2 n_i t + 3 \varepsilon - 2 \varepsilon_i - \varpi) \quad [9]$$

$$- m_i \frac{15}{16} \frac{a^3}{a_i^4} e \cos (4 n t - 3 n_i t + 4 \varepsilon - 3 \varepsilon_i - \varpi) \quad [10]$$

$$+ m_i \frac{9}{4} \frac{a^2}{a_i^3} e \cos (n t - 2 n_i t + \varepsilon - 2 \varepsilon_i + \varpi) \quad [12]$$

$$+ m_i \frac{45}{16} \frac{a^3}{a_i^4} e \cos (2 n t - 3 n_i t + 2 \varepsilon - 3 \varepsilon_i + \varpi) \quad [13]$$

$$- m_i \left\{ \frac{1}{a_i} + \frac{3}{4} \frac{a^2}{a_i^3} \right\} e_i \cos (n_i t + \varepsilon_i - \varpi_i) \quad [15]$$

$$- m_i \frac{3}{8} \frac{a^3}{a_i^4} e_i \cos (n t + \varepsilon_i - \varpi_i) \quad [16]$$

$$+ m_i \frac{3}{8} \frac{a^2}{a_i^3} e_i \cos (2 n t - n_i t + 2 \varepsilon - \varepsilon_i - \varpi_i) \quad [17]$$

$$+ m_i \frac{5}{8} \frac{a^3}{a_i^4} e_i \cos (3 n t - 2 n_i t + 3 \varepsilon - 2 \varepsilon_i - \varpi_i) \quad [18]$$

$$- m_i \frac{9}{8} \frac{a^3}{a_i^4} e_i \cos (n t - 2 n_i t + \varepsilon - 2 \varepsilon_i + \varpi_i) \quad [20]$$

$$- m_i \frac{21}{8} \frac{a^2}{a_i^3} e_i \cos (2 n t - 3 n_i t + 2 \varepsilon - 3 \varepsilon_i + \varpi_i) \quad [21]$$

$$- m_i \frac{25}{8} \frac{a^3}{a_i^4} e_i \cos (3 n t - 4 n_i t + 3 \varepsilon - 4 \varepsilon_i + \varpi_i) \quad [22]$$

$$- m_i \frac{15}{8} \frac{a^2}{a_i^3} e^2 \cos (2 n_i t + 2 \varepsilon_i - 2 \varpi) \quad [24]$$

$$- m_i \frac{33}{64} \frac{a^3}{a_i^4} e^2 \cos (n t + n_i t + \varepsilon + \varepsilon_i - 2 \varpi) \quad [25]$$

$$+ \frac{m_i}{8} \frac{a^2}{a_i^3} e^2 \cos (2 n t + 2 \varepsilon - 2 \varpi) \quad [26]$$

$$+ m_i \frac{9}{64} \frac{a^3}{a_i^4} e^2 \cos (3 n t - n_i t + 3 \varepsilon - \varepsilon_i - 2 \varpi) \quad [27]$$

$$-m_i \frac{3}{4} \frac{a^2}{a_i^3} e^2 \cos(4nt - 2n_it + 4\varepsilon - 2\varepsilon_i - 2\varpi) \quad [28]$$

Development of R according to powers of $\frac{a}{a_i}$.

$$-m_i \frac{75}{64} \frac{a^3}{a_i^4} e^2 \cos(5nt - 3n_it + 5\varepsilon - 3\varepsilon_i - 2\varpi) \quad [29]$$

$$-m_i \frac{285}{64} \frac{a^3}{a_i^4} e^2 \cos(nt - 3n_it + \varepsilon - 3\varepsilon_i + 2\varpi) \quad [32]$$

$$+m_i \frac{3}{4} \frac{a^2}{a_i^3} e e_i \cos(nt - n_it + \varepsilon - \varepsilon_i - \varpi + \varpi_i) \quad [35]$$

$$+m_i \frac{9}{16} \frac{a^3}{a_i^4} e e_i \cos(2nt - 2n_it + 2\varepsilon - 2\varepsilon_i - \varpi + \varpi_i) \quad [36]$$

$$-m_i \frac{21}{8} \frac{a^2}{a_i^3} e e_i \cos(3nt - 3n_it + 3\varepsilon - 3\varepsilon_i - \varpi + \varpi_i) \quad [37]$$

$$+m_i \frac{15}{16} \frac{a^3}{a_i^4} e e_i \cos(\varpi - \varpi_i) \quad [41]$$

$$-m_i \frac{9}{8} \frac{a^2}{a_i^3} e e_i \cos(nt - n_it + \varepsilon - \varepsilon_i + \varpi - \varpi_i) \quad [42]$$

$$-m_i \frac{45}{16} \frac{a^3}{a_i^4} e e_i \cos(2nt - 2n_it + 2\varepsilon - 2\varepsilon_i + \varpi - \varpi_i) \quad [43]$$

$$+m_i \frac{45}{16} \frac{a^3}{a_i^4} e e_i \cos(2n_it + 2\varepsilon_i - \varpi - \varpi_i) \quad [46]$$

$$+m_i \frac{3}{4} \frac{a^2}{a_i^3} e e_i \cos(nt + n_it + \varepsilon + \varepsilon_i - \varpi - \varpi_i) \quad [47]$$

$$+m_i \frac{3}{16} \frac{a^3}{a_i^4} e e_i \cos(2nt + 2\varepsilon - \varpi - \varpi_i) \quad [48]$$

$$+m_i \frac{3}{8} \frac{a^2}{a_i^3} e e_i \cos(3nt - n_it + 3\varepsilon - \varepsilon_i - \varpi - \varpi_i) \quad [49]$$

$$+m_i \frac{15}{16} \frac{a^3}{a_i^4} e e_i \cos(4nt - 2n_it + 4\varepsilon - 2\varepsilon_i - \varpi - \varpi_i) \quad [50]$$

$$+m_i \frac{63}{8} \frac{a^2}{a_i^3} e e_i \cos(nt - 3n_it + \varepsilon - 3\varepsilon_i + \varpi + \varpi_i) \quad [53]$$

$$+m_i \frac{225}{16} \frac{a^3}{a_i^4} e e_i \cos(2nt - 4n_it + 2\varepsilon - 4\varepsilon_i + \varpi + \varpi_i) \quad [54]$$

Development of R according to powers of $\frac{a}{a_i}$.

$$+ m_i \left\{ \frac{1}{a_i} - \frac{9}{8} \frac{a^2}{a_i^3} \right\} e_i^2 \cos(2 n_i t + 2 \varepsilon_i - 2 \varpi_i) \cdot \qquad [57]$$

$$- m_i \frac{33}{64} \frac{a^3}{a_i^4} e_i^2 \cos(n t + n_i t + \varepsilon + \varepsilon_i - 2 \varpi_i) \qquad [58]$$

$$- m_i \frac{5}{64} \frac{a^3}{a_i^4} e_i^2 \cos(3 n t - n_i t + 3 \varepsilon - \varepsilon_i - 2 \varpi_i) \qquad [60]$$

$$- m_i \frac{159}{64} \frac{a^3}{a_i^4} e_i^2 \cos(n t - 3 n_i t + \varepsilon - 3 \varepsilon_i + 2 \varpi_i) \qquad [63]$$

$$- m_i \frac{51}{8} \frac{a^3}{a_i^3} e_i^2 \cos(2 n t - 4 n_i t + 2 \varepsilon - 4 \varepsilon_i + 2 \varpi_i) \qquad [64]$$

$$- m_i \frac{635}{64} \frac{a^3}{a_i^4} e_i^2 \cos(3 n t - 5 n_i t + 3 \varepsilon - 5 \varepsilon_i + 2 \varpi_i) \qquad [65]$$

$$- m_i \frac{9}{4} \frac{a^3}{a_i^4} \sin^2 \frac{i}{2} \cos(n t + n_i t + \varepsilon + \varepsilon_i - 2 \nu) \qquad [75]$$

$$- m_i \frac{3}{2} \frac{a^3}{a_i^3} \sin^2 \frac{i}{2} \cos(2 n t + 2 \varepsilon - 2 \nu) \qquad [76]$$

$$- m_i \frac{3}{2} \frac{a^2}{a_i^3} \sin^2 \frac{i}{2} \cos(2 n_i t + 2 \varepsilon_i - 2 \nu) \qquad [77]$$

$$- m_i \frac{15}{8} \frac{a^3}{a_i^4} \sin^2 \frac{i}{2} \cos(3 n t - n_i t + 3 \varepsilon - \varepsilon_i - 2 \nu) \qquad [78]$$

$$- m_i \frac{15}{8} \frac{a^3}{a_i^4} \sin^2 \frac{i}{2} \cos(n t - 3 n_i t + \varepsilon - 3 \varepsilon_i + 2 \nu) \qquad [79]$$

If according to the notation of M. Damoiseau, (Théorie Lunaire, p. 547, Mémoires des Savans Étrangers,) $n t - n_i t + \varepsilon - \varepsilon_i = t$, and x and z be put for the mean anomalies of m and m_i respectively and y for the distance of the planet m from its node, or, what is the same in the Lunar Theory, the distance of the moon from her node reckoned on the ecliptic ($\iota_i = 0$),

$$R = m_i \Bigg\{ \underbrace{- \frac{1}{a_i} \left(1 + \frac{a^2}{4 a_i^2} \right) + \frac{3}{2} \left(\sin^2 \frac{i}{2} - \frac{(e^2 + e_i^2)}{4} \right) \frac{a^2}{a_i^3}}_{[0]}$$

$$+ \left\{ \left(\cos^2 \frac{i}{2} - \frac{(e^2 + e_i^2)}{2} \right) \frac{a}{a_i^2} - \frac{a}{a_i^2} \left(1 + \frac{3}{8} \frac{a^2}{a_i^2} \right) + \sin^2 \frac{i}{2} \frac{a}{a_i^2} \left(1 + \frac{33}{8} \frac{a^2}{a_i^2} \right) \right.$$

$$\left. + \frac{(e^2 + e_i^2)}{2} \frac{a}{a_i^2} \left(1 - \frac{3}{2} \frac{a^2}{a_i^2} \right) \right\} \underbrace{\cos t}_{1} + \underbrace{\left\{ - \frac{3}{4} \frac{a^2}{a_i^3} + \frac{3}{2} \sin^2 \frac{i}{2} \frac{a^2}{a_i^3} + \frac{15}{8} (e^2 + e_i^2) \frac{a^2}{a_i^3} \right\} \cos 2 t}_{[2]}$$

Development of R according to powers of $\frac{a}{a'}$.

$$+\left\{-\frac{5}{8}\frac{a^3}{a_i^4}+\frac{15}{8}\sin^2\frac{i}{2}\frac{a^3}{a_i^4}+\frac{15}{4}(e^2+e_i^2)\frac{a^3}{a_i^4}\right\}\cos 3t+\frac{15}{16}\frac{a^3}{a_i^4}\cos(t-x)$$
[3] [6]

$$+\frac{a^3}{2a_i^3}e\cos x+\frac{3}{16}\frac{a^3}{a_i^4}e\cos(t+x)-\frac{3}{4}\frac{a^3}{a_i^3}e\cos(2t+x)-\frac{15}{16}\frac{a^3}{a_i^4}e\cos(3t+x)$$
[7] [8] [9] [10]

$$+\frac{9}{4}\frac{a^3}{a_i^3}e\cos(2t-x)+\frac{45}{16}\frac{a^3}{a_i^4}e\cos(3t-x)-\left\{\frac{1}{a_i}+\frac{3}{4}\frac{a^2}{a_i^3}\right\}e_i\cos z$$
[12] [13] [15]

$$-\frac{3}{8}\frac{a^3}{a_i^4}e_i\cos(t+z)+\frac{3}{8}\frac{a^3}{a_i^3}e_i\cos(2t+z)+\frac{5}{8}\frac{a_i^3}{a_i^4}e_i\cos(3t+z)$$
[16] [17] [18]

$$-\frac{9}{8}\frac{a^3}{a_i^4}e_i\cos(t-z)-\frac{21}{8}\frac{a^3}{a_i^3}e_i\cos(2t-z)-\frac{25}{8}\frac{a^3}{a_i^4}e_i\cos(3t-z)$$
[20] [21] [22]

$$-\frac{15}{8}\frac{a^3}{a_i^3}e^2\cos(2t-2x)-\frac{33}{64}\frac{a^3}{a_i^4}e^2\cos(t-2x)+\frac{a^2}{8a_i^3}e^2\cos 2x$$
[24] [25] [26]

$$+\frac{9}{64}\frac{a^3}{a_i^4}e^2\cos(t+2x)-\frac{3}{4}\frac{a^3}{a_i^3}e^2\cos(2t+2x)-\frac{75}{64}\frac{a^3}{a_i^4}e^2\cos(3t+2x)$$
[27] [28] [29]

$$-\frac{285}{64}\frac{a^3}{a_i^4}e^2\cos(3t-2x)+\frac{3}{4}\frac{a^3}{a_i^3}ee_i\cos(x-z)+\frac{9}{16}\frac{a^3}{a_i^4}ee_i\cos(t-z+x)$$
[32] [35] [36]

$$-\frac{27}{8}\frac{a^3}{a_i^3}ee_i\cos(2t-z+x)+\frac{15}{16}\frac{a^3}{a_i^4}ee_i\cos(t+z-x)-\frac{9}{8}\frac{a^3}{a_i^3}ee_i\cos(2t+z-x)$$
[37] [41] [42]

$$-\frac{45}{16}\frac{a^3}{a_i^4}ee_i\cos(3t+z-x)+\frac{45}{16}\frac{a^3}{a_i^4}ee_i\cos(t-z-x)+\frac{3}{4}\frac{a^3}{a_i^3}ee_i\cos(x+z)$$
[43] [46] [47]

$$+\frac{3}{16}\frac{a^3}{a_i^4}ee_i\cos(t+z+x)+\frac{3}{8}\frac{a^3}{a_i^3}ee_i\cos(2t+z+x)+\frac{15}{16}\frac{a^3}{a_i^4}ee_i\cos(3t+z+x)$$
[48] [49] [50]

$$+\frac{63}{8}\frac{a^3}{a_i^3}ee_i\cos(2t-z-x)-\frac{225}{16}\frac{a^3}{a_i^4}ee_i\cos(3t-z-x)+\left\{\frac{1}{a_i}-\frac{9}{8}\frac{a^2}{a_i^3}\right\}e_i^2\cos 2z$$
[53] [54] [57]

$$-\frac{33}{64}\frac{a^3}{a_i^4}e_i^2\cos(t+2z)-\frac{5}{64}\frac{a^3}{a_i^4}e_i^2\cos(3t+2z)-\frac{159}{64}\frac{a^3}{a_i^4}e_i^2\cos(t-2z)$$
[58] [60] [63]

$$-\frac{51}{8}\frac{a^3}{a_i^3}e_i^2(2t-2z)-\frac{635}{64}\frac{a^3}{a_i^4}e_i^2\cos(3t-2z)-\frac{9}{4}\frac{a^3}{a_i^4}\sin^2\frac{i}{2}\cos(t-2y)$$
[64] [65] [75]

$$-\frac{3}{2}\frac{a^2}{a_i^3}\sin^2\frac{i}{2}\cos 2y \;[76] \;-\frac{3}{2}\frac{a^2}{a_i^3}\sin^2\frac{i}{2}\cos(2t-2y) \;[77] \;-\frac{15}{8}\frac{a^3}{a_i^4}\cos(t+2y) \;[78]$$

$$-\frac{15}{8}\frac{a^3}{a_i^4}\sin^2\frac{i}{2}\cos(3t-2y)\Big\} \;[79]$$

$$\text{Let } \frac{a}{r} = 1 + e\cos\Big(n(1+k)t + s - \varpi\Big) + e^2(1+r_{48})\cos\Big(2n(1+k_2)t + 2s - 2\varpi\Big)$$

$$+ e_i^2 r_{50}\cos\Big(2n(1+k_{12})t + 2s - 2\varpi_i\Big) + r_0 + r_1\cos(nt + n_i t + s - s_i)$$

$$+ r_2\cos(2nt - 2n_i t + 2s - 2s_i) + \&c. + e r_6\cos(n_i t + s_i - \varpi) + \&c.$$

$$\frac{d^2 r^3}{2\,dt^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int dR + r\left(\frac{dR}{dr}\right) = 0$$

Let $\delta.\frac{1}{r}$ denote that part of $\frac{1}{r}$ which depends on the first power of the disturbing force. It is more simple to obtain $\delta.\frac{1}{r}$ from the above differential equation than δr; and the circumstance that the elliptic value of r^3 does not contain any term $e^2\cos(2nt + 2s - 2\varpi)$, gives an additional facility.

$$-\frac{d^2.r^3\delta.\frac{1}{r}}{dt^2} - \mu\delta.\frac{1}{r} + 2\int dR + r\left(\frac{dR}{dr}\right) = 0$$

When the disturbing force is neglected

$$r^3 = a^3\Big\{1 + 3e^2\Big(1 + \frac{e^2}{12}\Big) - 3e\Big(1 + \frac{3}{8}e^2\Big)\cos(nt + s - \varpi) - \frac{3}{8}e^2\cos(2nt + 2s - 2\varpi)$$

$$+ \frac{e^3}{8}\cos(3nt + 3s - 3\varpi) + \frac{e^4}{8}\cos(4nt + 4s - 4\varpi)\Big\}$$

Integrating the above differential equation by the method of indeterminate co-efficients, q_n being the co-efficient of the nth argument in the development of $2\int dR + r\left(\frac{dR}{dr}\right)$,

$$-r_0 + \frac{m_i}{\mu}a\,q_0 = 0$$

$$\frac{(n-n_i)^2}{n^2}\Big\{(1+3e^2)r_1 - \frac{3e^2}{2}(r_6 + r_8)\Big\} - r_1 + \frac{m_i}{\mu}a\,q_1 = 0$$

$$\frac{(2n-2n_i)^2}{n^2}\Big\{(1+3e^2)r_2 - \frac{3e^2}{2}(r_9 + r_{12})\Big\} - r_2 + \frac{m_i}{\mu}a\,q_2 = 0$$

Equations which serve to determine the coefficients of the inequalities of the réciprocal of the radius vector.

$$\frac{(3n-3n_{,})^2}{n^2}\left\{(1+3e^2)r_3-\frac{3e^2}{2}(r_{10}+r_{13})\right\}-r_3+\frac{m_{,}}{\mu}aq_3=0$$

$$\frac{(4n-4n_{,})}{n^2}\left\{(1+3e^2)r_4-\frac{3e^2}{2}(r_{11}+r_{14})\right\}-r_4+\frac{m_{,}}{\mu}aq_4=0$$

$$\frac{(5n-5n_{,})^2}{n^2}(1+3e^2)r_5-r_5+\frac{m_{,}}{\mu}aq_5=0$$

$$\frac{n^2}{n_{,}^2}\left\{r_6-\frac{3}{2}r_1\right\}-r_6+\frac{m_{,}}{\mu}aq_6=0$$

$$*\frac{(2n-n_{,})^2}{n^2}\left\{r_8-\frac{3}{2}r_1\right\}-r_8+\frac{m_{,}}{\mu}aq_8=0$$

$$\frac{(3n-2n_{,})^2}{n^2}\left\{r_9-\frac{3}{2}r_2\right\}-r_9+\frac{m_{,}}{\mu}aq_9=0$$

$$\frac{(4n-3n_{,})^2}{n^2}\left\{r_{10}-\frac{3}{2}r_3\right\}-r_{10}+\frac{m_{,}}{\mu}aq_{10}=0$$

$$\frac{(5n-4n_{,})^2}{n^2}\left\{r_{11}-\frac{3}{2}r_4\right\}-r_{11}+\frac{m_{,}}{\mu}aq_{11}=0$$

$$\frac{(n-2n_{,})^2}{n^2}\left\{r_{12}-\frac{3}{2}r_2\right\}-r_{12}+\frac{m_{,}}{\mu}aq_{12}=0$$

$$\frac{(2n-3n_{,})^2}{n^2}\left\{r_{13}-\frac{3}{2}r_3\right\}-r_{13}+\frac{m_{,}}{\mu}aq_{13}=0$$

$$\frac{(3n-4n_{,})^2}{n^2}\left\{r_{14}-\frac{3}{2}r_4\right\}-r_{14}+\frac{m_{,}}{\mu}aq_{14}=0$$

$$\frac{n_{,}^2}{n^2}r_{15}-r_{15}+\frac{m_{,}}{\mu}aq_{15}=0$$

$$\frac{(2n-n_{,})^2}{n^2}r_{17}-r_{17}+\frac{m_{,}}{\mu}aq_{17}=0$$

$$\frac{(3n-2n_{,})^2}{n^2}r_{18}-r_{18}+\frac{m_{,}}{\mu}aq_{18}=0$$

$$\frac{(4n-3n_{,})^2}{n^2}r_{19}-r_{19}+\frac{m_{,}}{\mu}aq_{19}=0$$

$$\frac{(n-3n_{,})^2}{n^2}r_{20}-r_{20}+\frac{m_{,}}{\mu}aq_{20}-0$$

$$\frac{(2n-3n_{,})^2}{n^2}r_{21}-r_{21}+\frac{m_{,}}{\mu}aq_{21}=0$$

$$\frac{(3n-4n_{,})^2}{n^2}r_{22}-r_{22}+\frac{m_{,}}{\mu}aq_{22}=0$$

* For the determination of the quantity k, see p. 50.

Equations which serve to determine the coefficients of the inequalities of the reciprocal of the radius vector.

$$\frac{(4n-5n_1)^2}{n^2} r_{23} - r_{23} + \frac{m_1}{\mu} a q_{23} = 0$$

$$\frac{4n_1^2}{n^2}\left\{r_{24} - \frac{3}{2} r_{19}\right\} - r_{24} + \frac{m_1}{\mu} a q_{24} = 0$$

$$\frac{(n+n_1)^2}{n^2}\left\{r_{25} - \frac{3}{2} r_6\right\} - r_{25} + \frac{m_1}{\mu} a q_{25} = 0$$

$$\frac{(3n-n_1)^2}{n^2}\left\{r_{27} - \frac{3}{2} r_8\right\} - r_{27} + \frac{m_1}{\mu} a q_{27} = 0$$

$$\frac{(4n-2n_1)^2}{n^2}\left\{r_{28} - \frac{3}{2} r_9\right\} - r_{28} + \frac{m_1}{\mu} a q_{28} = 0$$

$$\frac{(5n-3n_1)^2}{n^2}\left\{r_{29} - \frac{3}{2} r_{10}\right\} - r_{29} + \frac{m_1}{\mu} a q_{29} = 0$$

$$\frac{(6n-4n_1)^2}{n^2}\left\{r_{30} - \frac{3}{2} r_{11}\right\} - r_{30} + \frac{m_1}{\mu} a q_{30} = 0$$

$$\frac{(7n-5n_1)^2}{n^2} r_{31} - r_{31} + \frac{m_1}{\mu} a q_{31} = 0$$

$$\frac{(n-3n_1)^2}{n^2}\left\{r_{32} - \frac{3}{2} r_{13}\right\} - r_{32} + \frac{m_1}{\mu} a q_{32} = 0$$

$$\frac{(2n-4n_1)^2}{n^2}\left\{r_{33} - \frac{3}{2} r_{14}\right\} - r_{33} + \frac{m_1}{\mu} a q_{33} = 0$$

$$\frac{(3n-5n_1)^2}{n^2} r_{34} - r_{34} + \frac{m_1}{\mu} a q_{34} = 0$$

$$\frac{(n-n_1)^2}{n^2}\left\{r_{35} - \frac{3}{2} r_{15}\right\} - r_{35} + \frac{m_1}{\mu} a q_{35} = 0$$

$$\frac{(2n-2n_1)^2}{n^2}\left\{r_{36} - \frac{3}{2} r_{20}\right\} - r_{36} + \frac{m_1}{\mu} a q_{36} = 0$$

$$\frac{(3n-3n_1)^2}{n^2}\left\{r_{37} - \frac{3}{2} r_{21}\right\} - r_{37} + \frac{m_1}{\mu} a q_{37} = 0$$

$$\frac{(4n-4n_1)^2}{n^2}\left\{r_{38} - \frac{3}{2} r_{22}\right\} - r_{38} + \frac{m_1}{\mu} a q_{38} = 0$$

$$\frac{(5n-5n_1)^2}{n^2}\left\{r_{39} - \frac{3}{2} r_{23}\right\} - r_{39} + \frac{m_1}{\mu} a q_{39} = 0$$

$$\frac{(6n-6n_1)^2}{n^2} r_{40} - r_{40} + \frac{m_1}{\mu} a q_{40} = 0$$

$$- r_{41} + \frac{m_1}{\mu} a q_{41} = 0$$

$$\frac{(n-n_1)^2}{n^2}\left\{r_{42} - \frac{3}{2} r_{17}\right\} - r_{42} + \frac{m_1}{\mu} a q_{42} = 0$$

Equations which serve to determine the coefficients of the inequalities of the reciprocal of the radius vector.

$$\frac{(2n-2n_{,})^2}{n^2}\left\{r_{43}-\frac{3}{2}r_{18}\right\}-r_{43}+\frac{m_{,}}{\mu}aq_{43}=0$$

$$\frac{(3n-3n_{,})^2}{n^2}\left\{r_{44}-\frac{3}{2}r_{19}\right\}-r_{44}+\frac{m_{,}}{\mu}aq_{44}=0$$

$$\frac{(4n-4n_{,})^2}{n^2}\left\{r_{45}-r_{45}+\frac{m_{,}}{\mu}aq_{45}=0\right.$$

$$\frac{4n_{,}^2}{n^2}\left\{r_{46}-\frac{3}{2}r_{20}\right\}-r_{46}+\frac{m_{,}}{\mu}aq_{46}=0$$

$$\frac{(n+n_{,})^2}{n^2}\left\{r_{47}-\frac{3}{2}r_{15}\right\}-r_{47}+\frac{m_{,}}{\mu}aq_{47}=0$$

$$\frac{(3n-3n_{,})^2}{n^2}\left\{r_{49}-\frac{3}{2}r_{17}\right\}-r_{49}+\frac{m_{,}}{\mu}aq_{43}=0$$

$$\frac{(4n-2n_{,})^2}{n^2}\left\{r_{50}-\frac{3}{2}r_{18}\right\}-r_{50}+\frac{m_{,}}{\mu}aq_{50}=0$$

$$\frac{(5n-5n_{,})^2}{n^2}\left\{r_{51}-\frac{3}{2}r_{19}\right\}-r_{51}+\frac{m_{,}}{\mu}aq_{51}=0$$

$$\frac{(6n-4n_{,})^2}{n^2}\left\{r_{52}-r_{52}+\frac{m_{,}}{\mu}aq_{52}=0\right.$$

$$\frac{(n-n_{,})^2}{n^2}\left\{r_{53}-\frac{3}{2}r_{21}\right\}-r_{53}+\frac{m_{,}}{\mu}aq_{53}=0$$

$$\frac{(2n-4n_{,})^2}{n}\left\{r_{54}-\frac{3}{2}r_{22}\right\}-r_{54}+\frac{m_{,}}{\mu}aq_{54}=0$$

$$\frac{(3n-5n_{,})^2}{n^2}\left\{r_{55}-\frac{3}{2}r_{23}\right\}-r_{55}+\frac{m_{,}}{\mu}aq_{55}=0$$

$$\frac{(4n-6n_{,})^2}{n^2}r_{56}-r_{56}+\frac{m_{,}}{\mu}aq_{56}=0$$

$$\frac{4n_{,}^2}{n^2}r_{57}-r_{57}+\frac{m_{,}}{\mu}aq_{57}=0$$

$$\frac{(n+n_{,})^2}{n^2}r_{58}-r_{58}+\frac{m_{,}}{\mu}aq_{58}=0$$

$$\frac{(3n-n_{,})^2}{n^2}r_{60}-r_{60}+\frac{m_{,}}{\mu}aq_{60}=0$$

$$\frac{(4n-2n_{,})^2}{n^2}r_{61}-r_{61}+\frac{m_{,}}{\mu}aq_{61}=0$$

$$\frac{(5n-3n_{,})^2}{n^2}r_{62}-r_{62}+\frac{m_{,}}{\mu}aq_{62}=0$$

$$\frac{(n-3n_{,})^2}{n^2}r_{63}-r_{63}+\frac{m_{,}}{\mu}aq_{63}=0$$

Equations which serve to determine the coefficients of the inequalities of the reciprocal of the radius vector.

$$\frac{(2n-4n_1)^2}{n^2} r_{64} - r_{64} + \frac{m_1}{\mu} a q_{64} = 0$$

$$\frac{(3n-5n_1)^2}{n^2} r_{65} - r_{65} + \frac{m_1}{\mu} a q_{65} = 0$$

$$\frac{(4n-6n_1)^2}{n^2} r_{66} - r_{66} + \frac{m_1}{\mu} a q_{66} = 0$$

$$\frac{(5n-7n_1)^2}{n^2} r_{67} - r_{67} + \frac{m_1}{\mu} a q_{67} = 0$$

$$\frac{(n+n_1)^2}{n^2} r_{68} - r_{68} + \frac{m_1}{\mu} a q_{68} = 0$$

In order to obtain the values of the coefficients of the inequalities from these equations when the cubes of the eccentricities are neglected, as has been the case throughout, the values of r_0, r_1, r_2, r_3, r_4, r_5, found from the first six equations by neglecting the terms multiplied by e^2, may be substituted in the succeeding equations, which will then serve to determine r_6, r_8, r_9, &c. and these values of r_6, r_8, r_9, &c. being substituted in the terms multiplied by e^2, of the equations which determine r_1, r_2, r_3, r_4, &c. more accurate values of those quantities may be obtained. All the other coefficients of which the general symbol is r with a numerical index at foot, may then be obtained in succession without any difficulty.

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{h}{r^2} - \frac{1}{r^2}\int \frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t$$

Let r denote the elliptic value of r, then

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{h}{r^2} + \frac{2h}{r}\delta.\frac{1}{r} - \frac{1}{r^2}\int \frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t$$

$$\frac{a^2}{r^2} = 1 + \frac{e^2}{2} + \frac{3}{8}e^4 + 2e\left(1 + \frac{3e^2}{8}\right)\cos(nt - \varpi) + \frac{5e^2}{2}\left(1 + \frac{2}{15}e^2\right)\cos(2nt - 2\varpi)$$
$$+ \frac{13}{4}e^3\cos(3nt - 3\varpi) + \frac{103}{24}e^4\cos(4nt - 4\varpi)$$

$$\frac{a}{r} = 1 + e\left(1 - \frac{e^2}{8}\right)\cos(nt - \varpi) + e^2\left(1 - \frac{e^2}{3}\right)\cos(2nt - 2\varpi)$$
$$+ \frac{9}{8}e^3\cos(3nt - 3\varpi) + \frac{4}{3}e^4\cos(4nt - 4\varpi)$$

Let R_n denote the coefficient of the cosine in the development of R which corresponds to the number n multiplied by e or e_1, &c. Thus

$$R_0 = \left\{ -\frac{3a}{a_1^2} + \frac{3a}{2a_1^2} b_{3,0} - \frac{a^2}{2a_1^3} b_{3,1} - \frac{a}{4a_1^2} b_{3,2} \right\}$$ See p. 31

Since $\frac{dR}{d\lambda} = \frac{dR}{ds}$, $\frac{dR}{ds}$ being the differential of R with respect to s, considering $s - \varpi$ and $s - \nu$ constant

$$\lambda = n \left\{ 1 + 2r_0 \right\} t + s$$

$$+ \left\{ 2 \left\{ r_1 \left(1 - \frac{e^2}{2} \right) + \frac{e^2}{2} (r_0 + r_2) \right\} - \frac{m_1}{\mu} \left\{ \left(1 + \frac{e^2}{2} \right) \frac{a R_1 n}{(n - n_1)} - \frac{e^2}{n_1} a R_0 n + \frac{e^2 a R_2 n}{(2n - n_1)} \right\} \right\}$$
$$\frac{n}{(n - n_1)} \sin (n t - n_1 t + s - s_1)$$

$$+ \left\{ 2 \left\{ r_2 \left(1 - \frac{e^2}{2} \right) + \frac{e^2}{2} (r_0 + r_{12}) \right\} - \frac{m_1}{\mu} \left\{ \left(1 + \frac{e^2}{2} \right) \frac{a R_2 n}{(n - n_1)} + \frac{2 e^2 a R_2 n}{(3n - 2n_1)} + \frac{2 e^2 a R_{12} n}{(n - 2n_1)} \right\} \right\}$$
$$\frac{n}{(2n - 2n_1)} \sin (2n t - 2n_1 t + 2s - 2s_1)$$

$$+ \left\{ 2 \left\{ r_3 \left(1 - \frac{e^2}{2} \right) + \frac{e^2}{2} (r_{10} + r_{13}) \right\} - \frac{m_1}{\mu} \left\{ \left(1 + \frac{e^2}{2} \right) \frac{a R_3 n}{(n - n_1)} + \frac{3 e^2 a R_{10} n}{(4n - 3n_1)} + \frac{3 e^2 a R_{13} n}{(2n - 3n_1)} \right\} \right\}$$
$$\frac{n}{(3n - 3n_1)} \sin (3n t - 3n_1 t + 3s - 3s_1)$$

$$+ \left\{ 2 \left\{ r_4 \left(1 - \frac{e^2}{2} \right) + \frac{e^2}{2} (r_{11} + r_{14}) \right\} - \frac{m_1}{\mu} \left\{ \left(1 + \frac{e^2}{2} \right) \frac{a R_4 n}{(n - n_1)} + \frac{4 e^2 a R_{11} n}{(5n - 4n_1)} + \frac{4 e^2 a R_{14} n}{(3n - 4n_1)} \right\} \right\}$$
$$\frac{n}{(4n - 4n_1)} \sin (4n t - 4n_1 t + 4s - 4s_1)$$

$$+ \left\{ 2 \left(r_0 + \frac{r_1}{2} \right) - \frac{m_1}{\mu} \left\{ -\frac{a R_0 n}{n_1} + \frac{a R_1 n}{(n - n_1)} \right\} \right\} \frac{e}{n_1} \sin (n_1 t + s_1 - \varpi)$$

$$+ \left\{ 2 \left(r_0 + \frac{r_1}{2} \right) - \frac{m_1}{\mu} \left\{ \frac{a R_0 n}{(2n - n_1)} + \frac{a R_1 n}{(n - n_1)} \right\} \right\} \frac{ne}{(n - n)} \sin (2n t - n_1 t + 2s - s_1 - \varpi)$$

$$+ \left\{ 2 \left(r_0 + \frac{r_2}{2} \right) - \frac{m_1}{\mu} \left\{ \frac{2 a R_0 n}{(3n - 2n_1)} + \frac{a R_2 n}{(n - n_1)} \right\} \right\} \frac{ne}{(3n - 2n_1)} \sin (3n t - 2n t + 3s - 2s_1 - \varpi)$$

$$+ \left\{ 2 \left(r_{10} + \frac{r_3}{2} \right) - \frac{m_1}{\mu} \left\{ \frac{3 a R_{10} n}{(4n - 3n_1)} + \frac{a R_3 n}{(n - n_1)} \right\} \right\} \frac{ne}{(4n - 3n_1)} \sin (4n t - 3n_1 t + 4s - 3s_1 - \varpi)$$

Expression for the longitude.

$$+\left\{2\left(r_{11}+\frac{r_4}{2}\right)-\frac{m_i}{\mu}\left\{\frac{4\,a\,R_{11}\,n}{(5n-4n_i)}+\frac{a\,R_4\,n}{(n-n_i)}\right\}\right\}\frac{n\,e}{(5n-4n_i)}\sin(5nt-4n_it+5\varepsilon-4\varepsilon_i-\varpi)$$

$$+\left\{2\left(r_{12}+\frac{r_3}{2}\right)-\frac{m_i}{\mu}\left\{\frac{2\,a\,R_{12}\,n}{(n-2n_i)}+\frac{a\,R_3\,n}{(n-n_i)}\right\}\right\}\frac{n\,e}{(n-2n_i)}\sin(nt-2n_it+\varepsilon-2\varepsilon_i+\varpi)$$

$$+\left\{2\left(r_{13}+\frac{r_2}{2}\right)-\frac{m_i}{\mu}\left\{\frac{3\,a\,R_{13}\,n}{(2n-3n_i)}+\frac{a\,R_2\,n}{(n-n_i)}\right\}\right\}\frac{n\,e}{(2n-3n_i)}\sin(2nt-3n_it+2\varepsilon-3\varepsilon_i+\varpi)$$

$$+\left\{2\left(r_{14}+\frac{r_4}{2}\right)-\frac{m_i}{\mu}\left\{\frac{4\,a\,R_{14}\,n}{(3n-4n_i)}+\frac{a\,R_4\,n}{(n-n_i)}\right\}\right\}\frac{n\,e}{(3n-4n_i)}\sin(3nt-4n_it+3\varepsilon-4\varepsilon_i+\varpi)$$

$$+2\,r_{15}\frac{n\,e_i}{n_i}\sin(n_it+\varepsilon_i-\varpi_i)$$

$$+\left\{2\,r_{17}-\frac{2\,m_i\,a\,R_{17}\,n}{\mu\,(2n-n_i)}\right\}\frac{n\,e_i}{(2n-n_i)}\sin(2nt-n_it+2\varepsilon-\varepsilon_i-\varpi_i)$$

$$+\left\{2\,r_{18}-\frac{3\,m_i\,a\,R_{18}\,n}{\mu\,(3n-2n_i)}\right\}\frac{n\,e_i}{(3n-2n_i)}\sin(3nt-2n_it+3\varepsilon-2\varepsilon_i-\varpi_i)$$

$$+\left\{2\,r_{19}-\frac{4\,a\,R_{19}\,n}{\mu\,(4n-3n_i)}\right\}\frac{n\,e_i}{(4n-3n_i)}\sin(4nt-3n_it+4\varepsilon-3\varepsilon_i-\varpi_i)$$

$$+\left\{2\,r_{20}-\frac{m_i\,a\,R_{20}\,n}{\mu\,(n-2n_i)}\right\}\frac{n\,e_i}{(n-2n_i)}\sin(nt-2n_it+\varepsilon-2\varepsilon_i+\varpi_i)$$

$$+\left\{2\,r_{21}-\frac{2\,m_i\,a\,R_{21}\,n}{\mu\,(2n-3n_i)}\right\}\frac{n\,e_i}{(2n-3n_i)}\sin(2nt-3n_it+2\varepsilon-2\varepsilon_i+\varpi_i)$$

$$+\left\{2\,r_{22}-\frac{3\,m_i\,a\,R_{22}\,n}{\mu\,(3n-4n_i)}\right\}\frac{n\,e_i}{(3n-4n_i)}\sin(3nt-4n_it+3\varepsilon-4\varepsilon_i+\varpi_i)$$

$$+\left\{2\,r_{23}-\frac{4\,m_i\,a\,R_{23}\,n}{\mu\,(4n-5n_i)}\right\}\frac{n\,e_i}{(4n-5n_i)}\sin(4nt-5n_it-4\varepsilon-5\varepsilon_i+\varpi_i)$$

$$+\left\{2\left(r_{24}+\frac{r_{19}}{2}\right)\frac{m_i}{\mu}\left\{\frac{a\,R_{24}\,n}{m_i}+\frac{2\,a\,R_{19}\,n}{(n-2n_i)}\right\}\right\}\frac{n\,e^2}{n_i}\sin(2n_it+2\varepsilon_i-2\varpi)$$

$$+\left\{2\left(r_{25}+\frac{r_6}{2}+\frac{r_i}{2}\right)-\frac{m_i}{\mu}\left\{\frac{a\,R_{25}\,n}{(n+n_i)}-\frac{a\,R_6\,n}{n_i}+\frac{5\,a\,R_1\,n}{4\,(n-n_i)}\right\}\right\}\frac{n\,e^2}{n_i}\sin(nt+n_it+\varepsilon+\varepsilon_i-2\varpi)$$

$$+\left\{2\left(r_{27}+\frac{r_5}{2}+\frac{r_i}{2}\right)-\frac{m_i}{\mu}\left\{\frac{a\,R_{27}\,n}{(3n-n_i)}+\frac{a\,R_5\,n}{(2n-n_i)}+\frac{5\,a\,R_1\,n}{4\,(n-n_i)}\right\}\right\}$$

$$\frac{n\,e^2}{(3n-n_i)}\sin(3nt-n_it+3\varepsilon-\varepsilon_i-2\varpi)$$

Expression for the longitude.

$$+\left\{2\left(r_{28}+\frac{r_9}{2}+\frac{r_3}{2}\right)-\frac{m_1}{\mu}\left\{\frac{2aR_{28}n}{(4n-2n_1)}+\frac{2aR_9n}{(3n-2n_1)}+\frac{5aR_3n}{4(n-n_1)}\right\}\right\}$$

$$\frac{ne^2}{(4n-2n_1)}\sin(4nt-2n_1t+4\varepsilon-2\varepsilon_1-2\varpi)$$

$$+\left\{2\left(r_{29}+\frac{r_{10}}{2}+\frac{r_3}{2}\right)-\frac{m_1}{\mu}\left\{\frac{3aR_{29}n}{(5n-3n_1)}+\frac{3aR_{10}n}{(4n-3n_1)}+\frac{5aR_3n}{4(n-n_1)}\right\}\right\}$$

$$\frac{ne^2}{(5n-3n_1)}\sin(5nt-3n_1t+5\varepsilon-3\varepsilon_1-2\varpi)$$

$$+\left\{2\left(r_{30}+\frac{r_{11}}{2}+\frac{r_4}{2}\right)-\frac{m_1}{\mu}\left\{\frac{4aR_{30}n}{(6n-4n_1)}+\frac{4aR_{11}n}{(5n-4n_1)}+\frac{5aR_4n}{4(n-n_1)}\right\}\right\}$$

$$\frac{ne^2}{(6n-4n_1)}\sin(6nt-4n_1t+6\varepsilon-4\varepsilon_1-2\varpi)$$

$$+\left\{2\left(r_{32}+\frac{r_{13}}{2}+\frac{r_3}{2}\right)-\frac{m_1}{\mu}\left\{\frac{3aR_{32}n}{(n-3n_1)}+\frac{3aR_{13}n}{(2n-3n_1)}+\frac{5aR_3n}{4(n-n_1)}\right\}\right\}$$

$$\frac{ne^2}{(n-3n_1)}\sin(nt-3n_1t+\varepsilon-3\varepsilon_1+2\varpi)$$

$$+\left\{2\left(r_{33}+\frac{r_{14}}{2}+\frac{r_4}{2}\right)-\frac{m_1}{\mu}\left\{\frac{4aR_{33}n}{(2n-4n_1)}+\frac{4aR_{14}n}{(3n-4n_1)}+\frac{5aR_4n}{4(n-n_1)}\right\}\right\}$$

$$\frac{ne^2}{(2n-4n_1)}\sin(2nt-4n_1t+2\varepsilon-4\varepsilon_1+2\varpi)$$

$$+\left\{2\left(r_{36}+\frac{r_{20}}{2}\right)-\frac{m_1}{\mu}\left\{\frac{aR_{36}n}{(2n-2n_1)}+\frac{aR_{20}n}{(n-2n_1)}\right\}\right\}\frac{nee_1}{(2n-2n_1)}\sin(2nt-2n_1t+2\varepsilon-2\varepsilon_1-\varpi+\varpi_1)$$

$$+\left\{2\left(r_{37}+\frac{r_{21}}{2}\right)-\frac{m_1}{\mu}\left\{\frac{2aR_{37}n}{(3n-3n_1)}+\frac{2aR_{21}n}{(2n-3n_1)}\right\}\right\}\frac{nee_1}{(3n-3n_1)}\sin(3nt-3n_1t+3\varepsilon-3\varepsilon_1-\varpi+\varpi_1)$$

$$+\left\{2\left(r_{38}+\frac{r_{22}}{2}\right)-\frac{m_1}{\mu}\left\{\frac{3aR_{38}n}{(4n-4n_1)}+\frac{3aR_{22}n}{(3n-4n_1)}\right\}\right\}\frac{nee_1}{(4n-4n_1)}\sin(4nt-4n_1t+4\varepsilon-4\varepsilon_1-\varpi+\varpi_1)$$

$$+\left\{2\left(r_{39}+\frac{r_{23}}{2}\right)-\frac{m_1}{\mu}\left\{\frac{4aR_{39}n}{(5n-5n_1)}+\frac{4aR_{23}n}{(4n-5n_1)}\right\}\right\}\frac{nee_1}{(5n-5n_1)}\sin(5nt-5n_1t+5\varepsilon-5\varepsilon_1-\varpi+\varpi_1)$$

$$+\left\{2\left(r_{43}+\frac{r_{18}}{2}\right)-\frac{m_1}{\mu}\left\{\frac{3aR_{43}n}{(2n-2n_1)}+\frac{3aR_{18}n}{(3n-2n_1)}\right\}\right\}\frac{nee_1}{(2n-2n_1)}\sin(2nt-2n_1t+2\varepsilon-2\varepsilon_1+\varpi-\varpi_1)$$

$$+\left\{2\left(r_{44}+\frac{r_{19}}{2}\right)-\frac{m_1}{\mu}\left\{\frac{4aR_{44}n}{(3n-3n_1)}+\frac{4aR_{19}n}{(4n-3n_1)}\right\}\right\}\frac{nee_1}{(3n-3n_1)}\sin(3nt-3n_1t+3\varepsilon-3\varepsilon_1+\varpi-\varpi_1)$$

$$+\left\{2\left(r_{46}+\frac{r_{20}}{2}\right)-\frac{m_1}{\mu}\left\{-\frac{aR_{46}n}{2n_1}+\frac{aR_{20}n}{(n-2n_1)}\right\}\right\}\frac{nee_1}{2n_1}\sin(2n_1t+2\varepsilon-\varpi-\varpi_1)$$

Expression for the longitude.

$$+\left\{2\left(r_{49}+\frac{r_{17}}{2}\right)-\frac{m_j}{\mu}\left\{\frac{2aR_{49}n}{(3n-n_j)}+\frac{2aR_{17}n}{(2n-n_j)}\right\}\right\}\frac{nee_j}{(3n-n_j)}\sin(3nt-n_jt+3s-s_j-\varpi-\varpi_j)$$

$$+\left\{2\left(r_{50}+\frac{r_{18}}{2}\right)-\frac{m_j}{\mu}\left\{\frac{3aR_{50}n}{(4n-2n_j)}+\frac{3aR_{18}n}{(3n-2n_j)}\right\}\right\}\frac{nee_j}{(4n-2n_j)}\sin(4nt-2n_jt+4s-2s_j-\varpi-\varpi_j)$$

$$+\left\{2\left(r_{51}+\frac{r_{19}}{2}\right)-\frac{m_j}{\mu}\left\{\frac{4aR_{51}n}{(5n-3n_j)}+\frac{4aR_{19}n}{(4n-3n_j)}\right\}\right\}\frac{nee_j}{(5n-3n_j)}\sin(5nt-3n_jt+5s-3s_j-\varpi-\varpi_j)$$

$$+\left\{2\left(r_{53}+\frac{r_{21}}{2}\right)-\frac{m_j}{\mu}\left\{\frac{2aR_{53}n}{(n-3n_j)}+\frac{2aR_{21}n}{(2n-3n_j)}\right\}\right\}\frac{nee_j}{(n-3n_j)}\sin(nt-3n_jt+s-3s_j+\varpi+\varpi_j)$$

$$+\left\{2\left(r_{54}+\frac{r_{22}}{2}\right)-\frac{m_j}{\mu}\left\{\frac{3aR_{54}n}{(2n-4n_j)}+\frac{3aR_{22}n}{(3n-4n_j)}\right\}\right\}\frac{nee_j}{(2n-4n_j)}\sin(2nt-4n_jt+2s-4s_j+\varpi+\varpi_j)$$

$$+\left\{2\left(r_{55}+\frac{r_{23}}{2}\right)-\frac{m_j}{\mu}\left\{\frac{4aR_{55}n}{(3n-5n_j)}+\frac{4aR_{23}n}{(4n-5n_j)}\right\}\right\}\frac{nee_j}{(3n-5n_j)}\sin(3nt-5n_jt+3s-5s_j+\varpi+\varpi_j)$$

$$+\left\{2r_{58}-\frac{m_j}{\mu}\frac{aR_{58}n}{(n+n_j)}\right\}\frac{ne_j^2}{(n+n_j)}\sin(nt+n_jt+s+s_j-2\varpi_j)$$

$$+\left\{2r_{59}-\frac{m_jaR_{59}}{\mu}\right\}e_j^2\sin(2nt+2s-2\varpi_j)$$

$$+\left\{2r_{60}-\frac{m_j}{\mu}\frac{3aR_{60}n}{(3n-n_j)}\right\}\frac{ne_j^2}{(3n-n_j)}\sin(3nt-n_jt+3s-s_j-2\varpi_j)$$

$$+\left\{2r_{61}-\frac{m_j}{\mu}\frac{4aR_{61}n}{(4n-2n_j)}\right\}\frac{ne_j^2}{(4n-2n_j)}\sin(4nt-2n_jt+4s-2s_j-2\varpi_j)$$

$$+\left\{2r_{62}-\frac{m_j}{\mu}\frac{5aR_{62}n}{(5n-3n_j)}\right\}\frac{ne_j^2}{(5n-3n_j)}\sin 5nt-3n_jt+5s-3s_j-2\varpi_j)$$

$$+\left\{2r_{63}-\frac{m_j}{\mu}\frac{aR_{63}n}{(n-3n_j)}\right\}\frac{ne_j^2}{(n-3n_j)}\sin(nt-3n_jt+s-3s_j+2\varpi_j)$$

$$+\left\{2r_{64}-\frac{m_j}{\mu}\frac{2aR_{64}n}{(2n-4n_j)}\right\}\frac{ne_j^2}{(2n-4n_j)}\sin(2nt-4n_jt+2s-4s_j+2\varpi_j)$$

$$+\left\{2r_{65}-\frac{m_j}{\mu}\frac{3aR_{65}n}{(3n-5n_j)}\right\}\frac{ne_j^2}{(3n-5n_j)}\sin(3nt-5n_jt+3s-5s_j+2\varpi_j)$$

$$+\left\{2r_{66}-\frac{m_j}{\mu}\frac{4aR_{66}n}{(4n-6n_j)}\right\}\frac{ne_j^2}{(4n-6n_j)}\sin(4nt-6n_jt+4s-6s_j+2\varpi_j)$$

$$+\left\{2r_{67}-\frac{m_j}{\mu}\frac{5aR_{67}n}{(5n-7n_j)}\right\}\frac{ne_j^2}{(5n-7n_j)}\sin(5nt-7n_jt+5s-7s_j+2\varpi_j)$$

In order to convert the coefficients of the inequalities of the longitude into sexagesimal seconds, they must be multiplied by $\frac{1296000}{2\pi}$, the logarithm of which number is 5·3144251, the corresponding logarithm for centesimal seconds is 5·8038801.

$$\frac{\mathrm{d}R}{\mathrm{d}z} = m_i \left\{ \frac{z_i}{r_i^3} + \frac{z - z_i}{\{r^2 - 2rr_i\cos(\lambda - \lambda_i) + r_i^2\}^{\frac{3}{2}}} \right\}$$

If $z = 0$, and the products $e\tan\imath$, $e_i\tan\imath_i$ be neglected,

$$\frac{\mathrm{d}R}{\mathrm{d}z} = \frac{m_i}{a_i^3}\tan\imath_i \sin(n_i t + \varepsilon_i - \nu_i)\{1 - b_{3,0} - b_{3,1}\cos(nt - n_i t + \varepsilon - \varepsilon_i) - b_{3,2}\cos(2nt - 2n_i t + 2\varepsilon - 2\varepsilon_i)\}$$

$$\frac{a^3\,\mathrm{d}^2 s}{\mathrm{d}t^2} + \mu s + \frac{m_i}{a_i^3}a^3\tan\imath_i \sin(n_i t + \varepsilon_i - \nu_i)\{1 - b_{3,0} - b_{3,1}\cos(nt - n_i t + \varepsilon - \varepsilon_i) - \&c.\} = 0$$

Expression for the tangent of the latitude.

$$s = -\frac{m_i}{\mu}\frac{n^2}{(n - n_i)(n + n_i)}\frac{a^3}{a_i^3}\tan\imath_i\{1 - b_{3,0}\}\sin(n_i t + \varepsilon_i - \nu_i)$$

$$+ i\sin\Big((1 + l)nt + \varepsilon - \nu_i\Big)$$

$$- \frac{m_i}{\mu}\frac{n^2}{4n_i(2n - 2n_i)}\frac{a^3}{a_i^3}\tan\imath_i b_{3,1}\sin(nt - 2n_i t + \varepsilon - 2\varepsilon_i + \nu_i)$$

$$- \frac{m_i}{\mu}\frac{n^2}{2(n - n_i)(3n - n_i)}\frac{a^3}{a_i^3}\tan\imath_i b_{3,2}\sin(2nt - n_i t + 2\varepsilon - \varepsilon_i - \nu_i)$$

$$+ \frac{m_i}{\mu}\frac{n^2}{2(n - 3n_i)(3n - 3n_i)}\frac{a^3}{a_i^3}\tan\imath_i b_{3,2}\sin(2nt - 3n_i t + 2\varepsilon - 3\varepsilon_i + \nu_i)$$

$$- \frac{m_i}{\mu}\frac{n^2}{2(2n - 2n_i)(4n - 2n_i)}\frac{a^3}{a_i^3}\tan\imath_i b_{3,3}\sin(3nt - 2n_i t + 3\varepsilon - 2\varepsilon_i - \nu_i)$$

$$+ \frac{m_i}{\mu}\frac{n^2}{2(2n - 4n_i)(4n - 4n_i)}\frac{a^3}{a_i^3}\tan\imath_i b_{3,4}\sin(3nt - 4n_i t + 3\varepsilon - 4\varepsilon_i + \nu_i)$$

$$- \frac{m_i}{\mu}\frac{n^2}{2(3n - 3n_i)(5n - 3n_i)}\frac{a^3}{a_i^3}\tan\imath_i b_{3,4}\sin(4nt \quad 3n_i t + 4\varepsilon - 3\varepsilon_i - \nu_i)$$

$$l(2 + l)\,\mathrm{i} = \frac{m}{2\mu_i}\frac{a^3}{a_i^3}\tan\imath_i b_{3,1}$$

$$R_6 = -\frac{3a}{2a_i^3} + \frac{3a}{2a_i^3}b_{3,0} - \frac{a^3}{2a_i^3}b_{3,1} - \frac{a}{4a_i^3}b_{3,2} \quad \text{(See p. 31.)}$$

$$q_6 = -\frac{3a}{2a_i^3} + \frac{3}{2}\frac{a}{a_i^3}b_{3,0} - \frac{a^3}{2a_i^3}b_{3,1} - \frac{a}{4a_i^3}b_{3,2} - \frac{3\cdot 3a^3}{2a_i^3}\left\{\frac{a}{a_i}b_{3,0} - \tfrac{1}{2}b_{3,1}\right\}$$

$$+ \frac{3}{2}\frac{a^3}{a_i^4}\left\{\frac{a}{a_i}b_{3,1} - b_{3,0} - \tfrac{1}{2}b_{3,2}\right\} + \frac{3}{4}\frac{a^3}{a_i^3}\left\{\frac{a}{a_i}b_{3,2} - \tfrac{1}{2}b_{3,1} - \tfrac{1}{2}b_{3,3}\right\}$$

$$q_4 = -\frac{3a}{2a_i^2} + \frac{3}{2}\frac{a}{a_i^2}b_{3,0} - \frac{a^2}{2a_i^3}b_{3,1} - \frac{a}{4a_i^2}b_{3,2} + \frac{3}{2}\frac{a^2}{a_i^3}\left\{\left(1+\frac{a^2}{a_i^2}\right)b_{3,1} - 2\frac{a}{a_i}b_{3,0} - \frac{a}{a_i^2}b_{3,2}\right\}$$
$$-\frac{3a^3}{a_i^4}\left\{b_{3,0} - \tfrac{1}{2}b_{3,2}\right\} + \frac{3}{8}\frac{a^2}{a_i^3}\left\{b_{3,1} - b_{3,3}\right\}$$
$$= -\frac{3a}{2a_i^2} + \frac{3}{2}\frac{a}{a_i^2}b_{3,0} - \frac{a^2}{2a_i^3}b_{3,1} - \frac{a}{4a_i^2}b_{3,2} + \frac{3}{2}\frac{a^2}{a_i^3}b_{3,1} - \frac{a^2}{a_i^3}b_{3,1} + \frac{2}{4}\frac{a}{a_i^2}b_{3,2}$$
$$= -\frac{3a}{2a_i^2} + \frac{3}{2}\frac{a}{a_i^2}b_{3,0} - \frac{2a^2}{2a_i^3}b_{3,1} - \frac{a}{4a_i^2}b_{3,2}$$

The quantities of which the general symbol is q, and which refer to the terms in the development of R multiplied by the eccentricities, admit of similar reductions; so that

$$q_7 = -\frac{a^2}{a_i^3}b_{3,0} + \frac{a}{a_i^2}b_{3,1} \qquad q_{16} = -\frac{a}{2a_i^2}b_{3,2}$$

Considering only the terms in $2\int dR + r\left(\frac{dR}{dr}\right)$, of which the arguments are $nt+\varepsilon-\varpi$, and $nt+\varepsilon-\varpi_i$

$$2\int dR + r\left(\frac{dR}{dr}\right) = m_i q_7 e\cos(nt+\varepsilon-\varpi) + m_i q_{16} e_i \cos(nt+\varepsilon-\varpi_i)$$
$$= m_i \mathrm{q}\cos(nt+\varepsilon-\varpi_1)$$

provided

$$\mathrm{q}\cos\varpi_1 = q_7 e\cos\varpi + q_{16} e_i \cos\varpi_i$$
$$\mathrm{q}\cos\varpi_1 = q_7 e\sin\varpi + q_{16} e_i \sin\varpi_i$$

And if

$$\frac{a}{r} = 1 + r_0 + e\cos\left(n(1+k)t + \varepsilon - \varpi_1\right) + \&c.$$

$$(1+k)^2(1-3r_0) - 1 + \frac{m_i a}{\mu e}\mathrm{q} = 0 \qquad k = \frac{3}{2}r_0 - \frac{m_i a}{2\mu e}\mathrm{q}$$

$$r_0 = \frac{m_i}{\mu}\left\{\frac{a^3}{a_i^3}b_{3,0} - \frac{a^2}{2a_i^2}b_{3,1}\right\}$$

$$\mathrm{q}^2 = q_7^2 e^2 + 2q_7 q_{16} e e_i \cos(\varpi-\varpi_i) + q_{16}^2 e_i^2$$

$$\mathrm{q} = q_7 e + q_{16} e_i \cos(\varpi-\varpi_i) \text{ nearly}$$

$$\frac{\mathrm{q}}{e} = -\frac{a^2}{a_i^3}b_{3,0} + \frac{a}{a_i^2}b_{3,1} - \frac{a}{2a_i^2}\frac{e_i}{e}b_{3,2}\cos(\varpi-\varpi_i)$$

$$k = \frac{m_i}{\mu}\left\{\frac{3a^3}{2a_i^3}b_{3,0} - \frac{3a^2}{4a_i^2}b_{3,1} + \frac{a^2}{2a_i^3}b_{3,0} - \frac{a^2}{2a_i^2}b_{3,1} + \frac{a^2}{4a_i^2}b_{3,2}\frac{e_i}{e}\cos(\varpi-\varpi_i)\right\}$$
$$= \frac{m_i}{\mu}\left\{\frac{2a^3}{a_i^3}b_{3,0} - \frac{5}{4}\frac{a^2}{a_i^2}b_{3,1} + \frac{a^2}{4a_i^2}b_{3,2}\frac{e_i}{e}\cos(\varpi-\varpi_i)\right\}$$

$$\frac{d\lambda}{dt} = n(1+2r_0)t + \varepsilon + 2(1+r_0)e\cos\left(n(1+k)t + \varepsilon - \varpi_1\right) - \frac{m_j}{\mu} a R_{16} e_j \cos(nt + \varepsilon - \varpi_j)$$

and neglecting the square of the disturbing force

$$\frac{d\lambda}{dt} = n(1+2r_0)t + \varepsilon + 2(1+r_0)e\cos\left(n(1+k)t + \varepsilon - \varpi_j\right) - \frac{m_j}{\mu} a R_{16} e_j \cos\left(n(1+k)t + \varepsilon - \varpi_j\right)$$

If

$$e(1+k)\cos\varpi_2 = (1+r_0)e\cos\varpi_1 - \frac{m_j}{2\mu} a R_{16} e_j \cos\varpi_j$$

$$e(1+k)\sin\varpi_2 = (1+r_0)e\sin\varpi_1 - \frac{m_j}{2\mu} a R_{16} e_j \sin\varpi_j$$

$$\lambda = n(1+2r_0)t + \varepsilon 2e\sin\left(n(1+k)t + \varepsilon - \varpi_2\right)$$

$$e(1+r_0)\cos\varpi_1 = e(1+k)\cos\varpi_2 + \frac{m_j}{2\mu} a R_{16} e_j \cos\varpi_j$$

$$e(1+r_0)\sin\varpi_1 = e(1+k)\sin\varpi_2 + \frac{m_j}{2\mu} a R_{16} e_j \sin\varpi_j$$

$$e(1+r_0) = e(1+k)\left\{1 + \frac{m_j}{2\mu} a R_{16}\frac{e_j}{e}\cos(\varpi_2 - \varpi_j)\right\}$$

$$\cos\varpi_1 = \cos\varpi_2\left\{1 - \frac{m_j}{2\mu} a R_{16}\frac{e_j}{e}\cos(\varpi_2 - \varpi_j)\right\} + \frac{m_j}{2\mu} a R_{16}\frac{e_j}{e}\cos\varpi_j$$

$$\sin\varpi_1 = \sin\varpi_2\left\{1 - \frac{m_j}{2\mu} a R_{16}\frac{e_j}{e}\cos(\varpi_2 - \varpi_j)\right\} + \frac{m_j}{2\mu} a R_{16}\frac{e_j}{e}\sin\varpi_j$$

$$\sin(\varpi_2 - \varpi_1) = \frac{m_j}{2\mu} a R_{16}\frac{e_j}{e}\cos\varpi_j,$$

therefore neglecting the square of the disturbing force

$$\frac{e}{a} = \frac{e}{a}\left\{1 + k - r_0 + \frac{m_j}{2\mu} a R_{16}\frac{e_j}{e}\cos(\varpi - \varpi_j)\right\}$$

$$= \frac{e}{a}\left\{1 + \frac{m_j}{\mu}\left\{\frac{2}{a_j^3}b_{3,0} - \frac{5}{4}\frac{a^2}{a_j^3}b_{3,1} - \frac{a^3}{a_j^3}b_{3,0} + \frac{a^2}{2a_j^3}b_{3,1}\right\} + \frac{m_j}{\mu}\left\{\frac{a^2}{4a_j^3}b_{3,2} + \frac{3}{4}\frac{a^2}{a_j^3}b_{3,0} - \frac{a}{4}b_{3,1} - \frac{a^2}{8a_j^3}b_{3,2}\right\}\frac{e_j}{e}\cos(\varpi - \varpi_j)\right\}$$

$$= \frac{e}{a}\left\{1 + \frac{m_j}{\mu}\left\{\frac{a^3}{a_j^3}b_{3,0} - \frac{3}{4}\frac{a^2}{a_j^3}b_{3,1} - \frac{a^3}{4a_j^3}b_{3,1}\frac{e_j}{e}\cos(\varpi - \varpi_j)\right\}\right\}$$

Let

$$n\left\{1+\frac{2m_j}{\mu}\left(\frac{a^2}{a_j^3}b_{3,0}-\frac{a^2}{2a_j^3}b_{3,1}\right)\right\}=n$$

$$n(1+k)=n\left\{1-\frac{m}{4\mu_j}\frac{a^2}{a_j^3}b_{3,1}+\frac{m_j}{4\mu}\frac{a^2}{a_j^2}\frac{e_j}{e}b_{3,2}\cos(\varpi-\varpi_j)\right\}$$

Let $\frac{\mu}{n^2}=n^2$, then $a^3=n^3\left\{1+\frac{4m_j}{\mu}\left(\frac{a^3}{a_j^3}b_{3,0}-\frac{a^2}{a_j^3}b_{3,1}\right)\right\}$

$$e=n\left\{1+\frac{4m_j}{3\mu}\left(\frac{a^3}{a_j^3}b_{3,0}-\frac{a^2}{a_j^3}b_{3,1}\right)\right\}$$

$$\lambda=nt+\varepsilon+2e\sin\left(n\left\{1-\frac{m_j}{4\mu}\frac{a^2}{a_j^2}b_{3,1}+\frac{m_j}{4\mu_j}\frac{a^2}{a_j^2}\frac{e_j}{e}b_{3,2}\cos(\varpi-\varpi_j)\right\}t+\varepsilon-\varpi_0\right)+\&c.$$

$$\frac{a}{r}=1-\frac{m_j}{3\mu}\left\{\frac{a^3}{a_j^3}b_{3,0}-\frac{a^2}{2a_j^3}b_{3,1}\right\}$$

$$+e\left\{1-\frac{m_j}{\mu}\left\{\frac{a^3}{a_j^3}b_{3,0}+\frac{a^2}{12a_j^2}b_{3,1}-\frac{a^2}{4a_j^3}b_{3,2}\frac{e_j}{e}\cos(\varpi-\varpi_j)\right\}\right\}$$

$$\cos\left(n\left\{1-\frac{m_j}{4\mu}\frac{a^2}{a_j^2}b_{3,1}+\frac{m_j}{4\mu}\frac{a^2}{a_j^2}\frac{e_j}{e}b_{3,2}\cos(\varpi-\varpi_j)\right\}t+\varepsilon-\varpi_0\right)$$

$$+\frac{m_j}{\mu}\left\{\frac{3}{4}\frac{a^2}{a_j^3}b_{3,0}-\frac{a}{4a_j}b_{3,1}-\frac{a^2}{8a_j^3}b_{3,2}\right\}e_j\cos(nt+\varepsilon-\varpi_j)$$

$$\frac{r}{a}=1+\frac{m_j}{3\mu}\left\{\frac{a^3}{a_j^3}b_{3,0}-\frac{a^2}{2a_j^3}b_{3,1}\right\}$$

$$-e\left\{1+\frac{m_j}{\mu}\left\{\frac{a^3}{a_j^3}b_{3,0}-\frac{5}{12}\frac{a^2}{a_j^2}b_{3,1}+\frac{a^2}{a_j^3}b_{3,2}\frac{e_j}{e}\cos(\varpi-\varpi_j)\right\}\right\}$$

$$\cos\left(n\left\{1-\frac{m_j}{4\mu}\frac{a^2}{a_j^2}b_{3,1}+\frac{m_j}{4\mu}\frac{a^2}{a_j^2}\frac{e_j}{e}b_{3,2}\cos(\varpi-\varpi_j)\right\}t+\varepsilon-\varpi_0\right)$$

$$-\frac{m_j}{\mu}\left\{\frac{3}{4}\frac{a^2}{a_j^3}b_{3,0}-\frac{a}{4a_j}b_{3,1}-\frac{a^2}{8a_j^3}b_{3,2}\right\}e_j\cos(nt+\varepsilon-\varpi_j)+\&c.$$

If

$$\frac{a}{r}=1+r_0+e(1+f)\cos\left(n(1+k^*)t+\varepsilon-\varpi\right)+e_jf_j\cos\left(n(1+k_j)t+\varepsilon-\varpi_j\right)+\&c.$$

$$(1+f)\{(1+k)^2(1-3r_0)-1\}+\frac{m_j}{\mu}aq_7=0$$

* This quantity k must not be confounded with the quantity k above.

whence neglecting k^2, kf, &c.

$$k = \frac{3}{2} r_0 - \frac{m_1}{2\mu} a q_7$$

$$= \frac{m_1}{\mu}\left\{\frac{a^2}{a_1^2} b_{3,0} - \frac{a^2}{2 a_1^2} b_{3,1}\right\}$$

similarly

$$f_i\{(1+k_i)^2 - 1\} + \frac{m_j}{\mu} a q_{16} = 0$$

$$k_i^2 + 2k_i + \frac{m_j}{\mu}\frac{a}{f_i} q_{16} = 0, \qquad q_{16} = -\frac{a}{2 a_i^2} b_{3,2}$$

$$k_i = -2 - \frac{m_j}{4\mu}\frac{a^3}{f_i a_i^2} b_{3,2} \qquad \text{or} + \frac{m_j}{4\mu}\frac{a^3}{a_i^2 f_i} b_{3,2}$$

$$\lambda = n\{1 + 2r_0\} t + s + \frac{2e(1+f)(1+r_0)}{(1+k)} \sin\left(n(1+k)t + s - \varpi\right)$$

$$+ 2e_i\left\{\frac{f_i}{1+k_i} - \frac{m_i a}{2\mu} R_{16}\right\} \sin\left(n(1+k_i)t + s - \varpi_i\right) + \text{\&c. nearly}$$

If

$$\frac{e(1+f)(1+r_0)}{1+k} = e$$

$$(1+f) = e(1+k-r_0)$$

If $\frac{f_i}{1+k_i} - \frac{m_i}{2\mu} a R_{16} = 0$ and if k_i^2 be neglected

$$f_i = \frac{m_j}{4\mu}\frac{a^3}{a_i^3} b_{3,2} + \frac{m_i}{2\mu} a R_{16}$$

If $n(1+2r_0) = n$, and $n^2 = \frac{\mu}{a^3}$

$$a = a\left\{1 + \frac{4}{3} r_0\right\}$$

$$\frac{a}{r} = 1 - \frac{1}{3} r_0 + e\left\{1 + k - \frac{7}{3} r_0\right\} \cos\left(n(1+k)t + s - \varpi\right) + e_i f_i \cos\left(n(1+k_i)t + s - \varpi_i\right)$$

$$= 1 - \frac{1}{3} r_0 + e\left\{1 - \frac{5}{6} r_0 - \frac{m_i}{2\mu} a q_7\right\} \cos\left(n(1 + \frac{1}{2} r_0 - \frac{m_j}{2\mu} a q_7)t + s - \varpi\right)$$

$$- e_i\left\{\frac{m_i}{2\mu} a q_{16} - \frac{m_i}{2\mu} a R_{16}\right\} \cos\left(n(1+k_i)t + s - \varpi_i\right)$$

$$\frac{r}{a} = 1 + \frac{1}{3} r_0 - e\left\{1 - \frac{1}{6} r_0 - \frac{m_i}{2\mu} a q_7\right\} \cos\left(n(1 - \frac{1}{2} r_0 - \frac{m_j}{2\mu} a q_7)t + s - \varpi\right)$$

$$+ e\left\{\frac{m_i}{2\mu} a q_{16} - \frac{m_i}{2\mu} a R_{16}\right\} \cos\left(n(1+k_i)t + s - \varpi_i\right)$$

$$\frac{r}{a}=1+\frac{1}{3}r_0-e\left\{1-\frac{1}{6}r_0-\frac{m_1}{2\mu}aq_7\right\}\cos(nt+\varepsilon-\varpi)+e_1\left\{\frac{m_1}{2\mu}aq_{16}-\frac{m_1}{2\mu}aR_{16}\right\}\cos(nt+\varepsilon-\varpi_1)$$
$$-e\left\{\frac{1}{2}r_0+\frac{m_1}{2\mu}aq_7\right\}nt\sin(nt+\varepsilon-\varpi)-e_1\frac{m_1}{2\mu}aq_{16}nt\sin(nt+\varepsilon-\varpi_1)$$

In the notation of the Mécanique Céleste

$$r_0=\frac{a^2}{2}\left(\frac{\mathrm{d}A^{(0)}}{\mathrm{d}a}\right)\qquad aq_7=-\frac{a^3}{2}\left(\frac{\mathrm{d}^2A^{(0)}}{\mathrm{d}a^2}\right)-\frac{3a^2}{2}\left(\frac{\mathrm{d}A^{(0)}}{\mathrm{d}a}\right)$$

$$aq_{16}=-\frac{1}{2}\left\{a^2a'\left(\frac{\mathrm{d}^2A^{(1)}}{\mathrm{d}a\,\mathrm{d}a'}\right)+2a^2\left(\frac{\mathrm{d}A^{(1)}}{\mathrm{d}a}\right)+2a'a\left(\frac{\mathrm{d}A^{(1)}}{\mathrm{d}a'}\right)+4aA^{(1)}\right\}$$
$$=-\frac{1}{2}\left\{2aA-2a^2\left(\frac{\mathrm{d}A^{(1)}}{\mathrm{d}a}\right)-a^2\left(\frac{\mathrm{d}^2A^{(1)}}{\mathrm{d}a^2}\right)\right\}$$

$$R_{16}=-\frac{1}{2}a'\left(\frac{\mathrm{d}A^{(1)}}{\mathrm{d}a'}\right)-A^{(1)}$$
$$=\frac{a}{2}\left(\frac{\mathrm{d}A^{(1)}}{\mathrm{d}a}\right)-\frac{1}{2}A^{(1)}$$

$$\frac{\mu}{6m_1}r_0+\frac{a}{2}q_7=\frac{a^2}{12}\left(\frac{\mathrm{d}A^{(0)}}{\mathrm{d}a}\right)-\frac{a^3}{4}\left(\frac{\mathrm{d}^2A^{(0)}}{\mathrm{d}a^2}\right)-\frac{3}{4}a^2\left(\frac{\mathrm{d}A^{(0)}}{\mathrm{d}a}\right)$$
$$=-\frac{2a^2}{3}\left(\frac{\mathrm{d}A^{(0)}}{\mathrm{d}a}\right)-\frac{a^3}{4}\left(\frac{\mathrm{d}^2A^{(0)}}{\mathrm{d}a^2}\right)=-f$$

$$\frac{a}{2}q_{16}-\frac{a}{2}R_{16}=-\frac{aA^{(1)}}{2}+\frac{a^2}{2}\left(\frac{\mathrm{d}A^{(1)}}{\mathrm{d}a}\right)+\frac{a^3}{4}\left(\frac{\mathrm{d}^2A^{(1)}}{\mathrm{d}a^2}\right)+\frac{aA^{(1)}}{4}-\frac{a^2}{4}\left(\frac{\mathrm{d}A^{(1)}}{\mathrm{d}a}\right)$$
$$=-\frac{aA^{(1)}}{4}+\frac{a^2}{4}\left(\frac{\mathrm{d}A^{(1)}}{\mathrm{d}a}\right)+\frac{a^3}{4}\left(\frac{\mathrm{d}^2A^{(1)}}{\mathrm{d}a^2}\right)=-f'$$

$$\frac{\mu}{2m_1}r_0+aq_7=\frac{a^2}{4}\left(\frac{\mathrm{d}A^{(0)}}{\mathrm{d}a}\right)-\frac{a^3}{4}\left(\frac{\mathrm{d}^2A^{(0)}}{\mathrm{d}a^2}\right)-\frac{3}{4}a^2\left(\frac{\mathrm{d}A^{(0)}}{\mathrm{d}a}\right)$$
$$=-\frac{a^2}{2}\left(\frac{\mathrm{d}A^{(0)}}{\mathrm{d}a}\right)-\frac{a^3}{4}\left(\frac{\mathrm{d}^2A^{(0)}}{\mathrm{d}a^2}\right)=-\frac{C}{2}=\frac{a^2}{4a_1^2}b_{3,1}$$

$$-\frac{1}{2}aq_{16}=\frac{1}{2}\left\{aA-a^2\left(\frac{\mathrm{d}A^{(1)}}{\mathrm{d}a}\right)-\frac{a^3}{2}\left(\frac{\mathrm{d}^2A^{(1)}}{\mathrm{d}a^2}\right)\right\}=-\frac{D}{2}=\frac{a^2}{2a_1^2}b_{3,2}$$

These results evidently agree with those given in the Mécanique Céleste, vol. i. p. 279, with the exception of the sign in the value of f' marked with an asterisk, which I think requires alteration in that work.

Finally, neglecting the quantities multiplied by t, which may be made to de-

pend upon the secular inequalities of the constants ε, ϖ, &c., and the squares of the eccentricities.

$$\frac{r}{a} = 1 + \frac{m_1}{3\mu}\left\{\frac{a^2}{a_1^2}b_{3,0} - \frac{a^2}{2a_1^2}b_{3,1}\right\} - e\left\{1 + \frac{m_1}{\mu}\left\{\frac{a^3}{a_1^3}b_{3,0} - \frac{5}{12}\frac{a^2}{a_1^2}b_{3,1}\right\}\right\}\cos(nt+\varepsilon-\varpi)$$

$$-\left\{\frac{3}{4}\frac{a^2}{a_1^2}b_{3,0} - \frac{a}{4a_1}b_{3,1} + \frac{a^2}{8a_1^2}b_{3,2}\right\}e_1\cos(nt+\varepsilon-\varpi_1)$$

$$-r_1\cos(nt-n_1t+\varepsilon-\varepsilon_1) - r_2\cos(2nt-2n_1t+2\varepsilon-2\varepsilon_1) - r_3\cos(3nt-3n_1t+3\varepsilon-3\varepsilon_1) - \&c.$$

$$-\{r_8-r_1\}e\cos(n_1t+\varepsilon_1-\varpi) - \{r_9-r_1\}e\cos(2nt-n_1t+2\varepsilon-\varepsilon_1-\varpi)$$

$$-\{r_9+r_2\}e\cos(3nt-2n_1t+3\varepsilon-2\varepsilon_1-\varpi)$$

$$-\{r_{10}-r_3\}e\cos(4nt-3n_1t+4\varepsilon-3\varepsilon_1-\varpi) - \{r_{11}-r_4\}e\cos(5nt-4n_1t+5\varepsilon-4\varepsilon_1-\varpi)$$

$$-\{r_{12}-r_5\}e\cos(nt-2n_1t+\varepsilon-2\varepsilon_1+\varpi) - \{r_{13}-r_3\}e\cos(2nt-3n_1t+2\varepsilon-3\varepsilon_1+\varpi)$$

$$-\{r_{14}-r_4\}e\cos(3nt-4n_1t+3\varepsilon-4\varepsilon_1+\varpi) - r_{16}e_1\cos(n_1t+\varepsilon_1-\varpi_1) - r_{17}e_1\cos(nt+\varepsilon-\varpi_1) - \&c.$$

The constant part of R

$$= m_1\left\{-\frac{b_{1,0}}{a_1} + \frac{a}{2a_1^2}\left(\sin^2\frac{\iota}{2} - \frac{e^2+e_1^2}{4}\right)b_{3,1} + \frac{a^2}{4a_1^3}b_{3,2}\cos(\varpi-\varpi_1)\right\}$$ See p. 29.

If this quantity $= -F$ according to the notation of the Théor. Anal. vol. i. p. 336.

$$\mathrm{d}\varepsilon = (1-\sqrt{1-e^2})\,\mathrm{d}\varpi - \frac{2a^2n}{\mu}\left(\frac{\mathrm{d}F}{\mathrm{d}a}\right)\mathrm{d}t$$

$$\mathrm{d}\varpi = an\frac{\sqrt{1-e^2}}{\mu e}\left(\frac{\mathrm{d}F}{\mathrm{d}e}\right)\mathrm{d}t$$

$$\frac{\mathrm{d}\varepsilon - \mathrm{d}\varpi}{\mathrm{d}t} = \frac{m_1}{\mu}\left\{\frac{2a^2}{a_1^2}b_{3,0} - \frac{5}{4}\frac{a^2}{a_1^2}b_{3,1} + \frac{a^3}{4a_1^3}b_{3,2}\frac{e_1}{e}\cos(\varpi - \varpi_1)\right\}$$

Let, as hitherto, (Phil. Trans. for 1830. p. 336.)

$$r = \frac{h^2\sqrt{1+s^2}}{\mu\cos\iota^2\{\sqrt{1+s^2}+e\cos(\lambda-\varpi)} = a\{1-e'\cos(v-a)\}$$

Fig. 1.

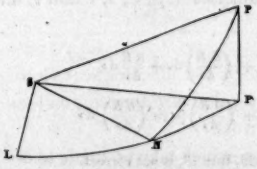
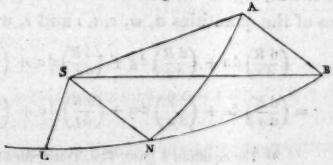


Fig. 2.



Let P be the place of the planet, P' its projection on the fixed plane L N P' (fig. 1 & 2.) S N the line of nodes, S L the line from which longitudes are reckoned. The angle L S P' $= \lambda'$. Let S A be the line of apsides. (fig. 2.)

In the notation of M. de Pontécoulant, vol. i. p. 316, the angle A S N $= g$, $\frac{s-\varpi}{n} = l$, M. de Pontécoulant has given expressions for the variations of the constants a, g, e', l, ι and ν in terms of the partial differences of the quantity R with regard to these quantities. It is easy from these to find similar expressions for the variations or differentials with regard to the time of the constants a, ϖ, e, ε, ι and ν.

Let S A B be a plane cutting the plane of the orbit at right angles, so that the angle S A B $= 90°$, A N B $= \iota$, B S N $= \varpi - \nu$

$$\frac{\mathrm{d}r^2}{\mathrm{d}t^2} + \frac{h^2}{r^2\cos\iota^2} - \frac{2\mu}{r} + \frac{\mu}{a} = 0$$

$$r = a\{1 - e'\cos(v - \alpha)\}$$

When r is a maximum or minimum $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$,

$$\frac{a h^2}{\mu\cos\iota^2} - 2ar + r^2 = 0, \quad \text{whence } r = a \pm \sqrt{a - \frac{h^2}{\mu\cos\iota^2}}$$

$$r = a(1 \pm e')$$

$$\frac{h^2}{\mu\cos\iota^2} = a(1 - e'^2)$$

By the equation of p. 336, line 12, (Phil. Trans. 1830.)

$$\frac{h^2}{\mu\cos\iota^2} = a\left(1 - e^2 + e^2\sin^2\iota\sin^2(\nu - \varpi)\right)$$

$$e'^2 = e^2\{1 - \sin^2\iota\sin^2(\nu - \varpi)\} * = e^2\cos^2 \mathrm{A S B}$$

Considering R first as a function of the quantities a, g, e', l, ι and ν, and then of the quantities a, ϖ, e, ε, ι and ν, we have

$$\left(\frac{\mathrm{d}R}{\mathrm{d}a}\right)\mathrm{d}a + \left(\frac{\mathrm{d}R}{\mathrm{d}g}\right)\mathrm{d}g + \left(\frac{\mathrm{d}R}{\mathrm{d}e'}\right)\mathrm{d}e' + \left(\frac{\mathrm{d}R}{\mathrm{d}l}\right)\mathrm{d}l + \left(\frac{\mathrm{d}R}{\mathrm{d}\iota}\right)\mathrm{d}\iota + \frac{\mathrm{d}R}{\mathrm{d}\nu}\mathrm{d}\nu$$

$$= \left(\frac{\mathrm{d}R}{\mathrm{d}a}\right)\mathrm{d}a + \left(\frac{\mathrm{d}R}{\mathrm{d}\varpi}\right)\mathrm{d}\varpi + \left(\frac{\mathrm{d}R}{\mathrm{d}e}\right)\mathrm{d}e + \left(\frac{\mathrm{d}R}{\mathrm{d}\varepsilon}\right)\mathrm{d}\varepsilon + \left(\frac{\mathrm{d}R}{\mathrm{d}\iota}\right)\mathrm{d}\iota + \left(\frac{\mathrm{d}R}{\mathrm{d}\nu}\right)\mathrm{d}\nu$$

* The equation I gave, Phil. Trans. for 1830, p. 336, line 17, is not correct.

By means of this equation, the equations

$$\varepsilon - \varpi = n\,l, \quad e'^2 = e^2\{1 - \sin^2 i \sin^2 (\nu - \varpi)\}$$

and the equations given by M. DE PONTÉCOULANT, vol. i. p. 328, the values of $\frac{\mathrm{d}a}{\mathrm{d}t}$, $\frac{\mathrm{d}\varpi}{\mathrm{d}t}$, $\frac{\mathrm{d}e}{\mathrm{d}t}$, $\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$, $\frac{\mathrm{d}i}{\mathrm{d}t}$ and $\frac{\mathrm{d}\nu}{\mathrm{d}t}$ may be easily obtained in terms of the quantities a, ϖ, e, ε, i, ν, and the partial differential coefficients $\left(\frac{\mathrm{d}R}{\mathrm{d}a}\right)$, $\left(\frac{\mathrm{d}R}{\mathrm{d}\varpi}\right)$, $\left(\frac{\mathrm{d}R}{\mathrm{d}e}\right)$, $\left(\frac{\mathrm{d}R}{\mathrm{d}\varepsilon}\right)$, $\left(\frac{\mathrm{d}R}{\mathrm{d}i}\right)$ and $\left(\frac{\mathrm{d}R}{\mathrm{d}\nu}\right)$.

Substituting in the equations of p. 40, for q_s their values and neglecting e^2, e_i^2, $e\,e_i$, and $\sin^2\frac{i_i}{2}$, when a is less than a_i;

Expression for the reciprocal of the radius vector, when $a < a_i$.

$$\frac{a}{r} = 1 - \frac{m_i}{\mu}\left\{\frac{a^3}{a_i^3}\,b_{3,0} - \frac{a^2}{a_i^2}\,b_{3,1}\right\} + e\left\{1 - \frac{m_i}{\mu}\left(\frac{a^3}{a_i^3}\,b_{3,0} + \frac{a^2}{12\,a_i^2}\,b_{3,1}\right)\right\}\cos(n\,t + \varepsilon - \varpi)$$

$$+ \frac{m_i}{\mu}\,\frac{n^2}{(2n - n_i)\,n_i}\left\{\frac{2n}{(n - n_i)}\left(\frac{a^2}{a_i^2} - \frac{a}{a_i}\,b_{1,1}\right) + \frac{a^2}{a_i^2}\right.$$

$$\left. + \frac{a^2}{a_i^2}\left(\frac{a}{a_i}\,b_{3,1} - b_{3,0} - \tfrac{1}{2}\,b_{3,2}\right)\right\}\cos(n\,t - n_i\,t + \varepsilon - \varepsilon_i) \qquad [1]$$

$$+ \frac{m_i}{\mu}\,\frac{n^2}{(3n - 2n_i)\,(n - 2n_i)}\left\{\frac{2n}{(n - n_i)}\,\frac{a}{a_i}\,b_{1,2}\right.$$

$$\left. - \frac{a^2}{a_i^2}\left(\frac{a}{a_i}\,b_{3,2} - \tfrac{1}{2}\,b_{3,1} - \tfrac{1}{2}\,b_{3,3}\right)\right\}\cos(2n\,t - 2n_i\,t + 2\varepsilon - 2\varepsilon_i) \qquad [2]$$

$$+ \frac{m_i}{\mu}\,\frac{n^2}{(4n - 3n_i)\,(2n - 3n_i)}\left\{\frac{2n}{(n - n_i)}\,\frac{a}{a_i}\,b_{1,3}\right.$$

$$\left. - \frac{a^2}{a_i^2}\left(\frac{a}{a_i}\,b_{3,3} - \tfrac{1}{2}\,b_{3,2} - b_{3,4}\right)\right\}\cos(3n\,t - 3n_i\,t + 3\varepsilon - 3\varepsilon_i) \qquad [3]$$

$$+ \frac{m_i}{\mu}\,\frac{n^2}{(5n - 4n_i)\,(3n - 4n_i)}\left\{\frac{2n}{(n - n_i)}\,\frac{a}{a_i}\,b_{1,4}\right.$$

$$\left. - \frac{a^2}{a_i^2}\left(\frac{a}{a_i}\,b_{3,4} - \tfrac{1}{2}\,b_{3,3} - \tfrac{1}{2}\,b_{3,5}\right)\right\}\cos(4n\,t - 4n_i\,t + 4\varepsilon + 4\varepsilon_i) \qquad [4]$$

$$+ \frac{m_i}{\mu}\,\frac{n^2}{(6n - 5n_i)\,(4n - 5n_i)}\left\{\frac{2n}{(n - n_i)}\,\frac{a}{a_i}\,b_{1,5}\right.$$

$$\left. - \frac{a^2}{a_i^2}\left(\frac{a}{a_i}\,b_{3,5} - \tfrac{1}{2}\,b_{3,4} - \tfrac{1}{2}\,b_{3,6}\right)\right\}\cos(5n\,t - 5n_i\,t + 5\varepsilon - 5\varepsilon_i) \qquad [5]$$

* In the terms multiplied by $\frac{m_i}{\mu}$ the quantities a & a, e & e may be used indifferently.

Expression for the reciprocal of the radius vector, when $a < a_1$.

$$-\frac{n^2}{(n-n_1)(n+n_1)}\left\{\frac{3n_1^2}{2n^2}r_1+\frac{m_1}{\mu}\left\{\frac{3a^2}{2a_1^2}-\frac{3a^2}{2a_1^2}b_{3,0}+\frac{a^2}{2a_1^2}b_{3,1}-\frac{a^2}{4a_1^2}b_{3,2}\right\}\right\}e\cos(n_1t+\varepsilon_1-\varpi) \quad [6]$$

$$+\frac{n^2}{(3n-n_1)(n-n_1)}\left\{\frac{3(2n-n_1)^2}{2n^2}r_1-\frac{m_1}{\mu}\left\{\frac{4n}{(2n-n_1)}\left\{\frac{a^2}{2a_1^2}-\frac{a^2}{2a_1^2}b_{3,0}-\frac{a^2}{2a_1^2}b_{3,1}+\frac{3a^2}{4a_1^2}b_{3,2}\right\}\right.\right.$$

$$\left.\left.+\frac{a^2}{2a_1^2}-\frac{a^2}{2a_1^2}b_{3,0}+\frac{3a^2}{2a_1^2}b_{3,1}-\frac{3a^2}{4a_1^2}b_{3,2}\right\}\right\}e\cos(2nt-n_1t+2\varepsilon-\varepsilon_1-\varpi) \quad [8]$$

$$+\frac{n^2}{(4n-2n_1)(2n-2n_1)}\left\{\frac{3(3n-2n_1)^2}{2n^2}r_2-\frac{m_1}{\mu}\left\{\frac{6n}{(3n-2n_1)}\left\{-\frac{a^2}{4a_1^2}b_{3,1}-\frac{a^2}{2a_1^2}b_{3,2}+\frac{3a^2}{4a_1^2}b_{3,3}\right\}\right.\right.$$

$$\left.\left.-\frac{a^2}{2a_1^2}b_{3,1}+\frac{5}{2}\frac{a^2}{a_1^2}b_{3,2}-\frac{3}{2}\frac{a^2}{a_1^2}b_{3,3}\right\}\right\}e\cos(3nt-2n_1t+3\varepsilon-2\varepsilon_1-\varpi) \quad [9]$$

$$+\frac{n^2}{(5n-3n_1)(3n-3n_1)}\left\{\frac{3(4n-3n_1)^2}{2n^2}r_3-\frac{m_1}{\mu}\left\{\frac{8n}{(4n-3n_1)}\left\{-\frac{a^2}{4a_1^2}b_{3,2}-\frac{a^2}{2a_1^2}b_{3,3}+\frac{3a^2}{4a_1^2}b_{3,4}\right\}\right.\right.$$

$$\left.\left.-\frac{3a^2}{4a_1^2}b_{3,2}+\frac{7}{2}\frac{a^2}{a_1^2}b_{3,3}-\frac{9}{4}\frac{a^2}{a_1^2}b_{3,4}\right\}\right\}e\cos(4nt-3n_1t+4\varepsilon-3\varepsilon_1-\varpi) \quad [10]$$

$$+\frac{n^2}{(6n-4n_1)(4n-4n_1)}\left\{\frac{3(5n-4n_1)^2}{2n^2}r_4-\frac{m_1}{\mu}\left\{\frac{10n}{(5n-4n_1)}\left\{-\frac{a^2}{4a_1^2}b_{3,3}-\frac{a^2}{2a_1^2}b_{3,4}+\frac{3a^2}{4a_1^2}b_{3,5}\right\}\right.\right.$$

$$\left.\left.-\frac{a^2}{a_1^2}b_{3,3}+\frac{9}{2}\frac{a^2}{a_1^2}b_{3,4}-\frac{3a^2}{a_1^2}b_{3,5}\right\}\right\}e\cos(5nt-4n_1t+5\varepsilon-4\varepsilon_1-\varpi) \quad [11]$$

$$-\frac{n^2}{2n_1(2n-2n_1)}\left\{\frac{3(n-2n_1)^2}{2n^2}r_2-\frac{m_1}{\mu}\left\{\frac{2n}{(n-2n_1)}\left\{\frac{3a^2}{4a_1^2}b_{3,1}-\frac{a^2}{2a_1^2}b_{3,2}-\frac{a^2}{4a_1^2}b_{3,3}\right\}\right.\right.$$

$$\left.\left.+\frac{3a^2}{2a_1^2}b_{3,1}-\frac{a^2}{a_1^2}b_{3,2}+\frac{a^2}{2a_1^2}b_{3,3}\right\}\right\}e\cos(nt-2n_1t+\varepsilon-2\varepsilon_1+\varpi) \quad [12]$$

$$+\frac{n^2}{(n-3n_1)(3n-3n_1)}\left\{\frac{3(2n-3n_1)^2}{2n^2}r_3-\frac{m_1}{\mu}\left\{\frac{4n}{(2n-3n_1)}\left\{\frac{3a^2}{4a_1^2}b_{3,2}-\frac{a^2}{2a_1^2}b_{3,3}-\frac{a^2}{4a_1^2}b_{3,4}\right\}\right.\right.$$

$$\left.\left.+\frac{9a^2}{4a_1^2}b_{3,2}-2\frac{a^2}{a_1^2}b_{3,3}+\frac{3a^2}{4a_1^2}b_{3,4}\right\}\right\}e\cos(2nt-3n_1t+2\varepsilon-3\varepsilon_1+\varpi) \quad [13]$$

$$+\frac{n^2}{(2n-4n_1)(4n-4n_1)}\left\{\frac{3(3n-4n_1)^2}{2n^2}r_4-\frac{m_1}{\mu}\left\{\frac{6n}{(3n-4n_1)}\left\{\frac{3a^2}{4a_1^2}b_{3,3}-\frac{a^2}{2a_1^2}b_{3,4}-\frac{a^2}{4a_1^2}b_{3,5}\right\}\right.\right.$$

$$\left.\left.+3\frac{a^2}{a_1^2}b_{3,3}-3\frac{a^2}{a_1^2}b_{3,4}+\frac{a^2}{a_1^2}b_{3,5}\right\}\right\}e\cos(3nt-4n_1t+3\varepsilon-4\varepsilon_1+\varpi) \quad [14]$$

$$-\frac{m_j}{\mu}\frac{n^2}{(n-n_j)(n+n_j)}\frac{a^2}{2a_j^2}b_{3,1}e_j\cos(n_jt+\varepsilon_j-\varpi_j) \qquad [15]$$

Expression for the reciprocal of the radius vector, when $a < a_j$.

$$-\frac{m_j}{\mu}\frac{n^2}{(n-n_j)(3n-n_j)}\left\{\frac{4n}{(2n-n_j)}\left\{\frac{3a^2}{4a_j^2}b_{3,1}-\frac{a}{2a_j}b_{3,2}-\frac{a^2}{4a_j^2}b_{3,3}\right\}\right.$$

$$\left.-\frac{9}{4}\frac{a^2}{a_j^2}b_{3,1}+\frac{2a}{a_j}b_{3,2}-\frac{a^2}{4a_j^2}b_{3,3}\right\}e_j\cos(2nt-n_jt+2\varepsilon-\varepsilon_j-\varpi_j) \qquad [17]$$

$$-\frac{m_j}{\mu}\frac{n^2}{(2n-n_j)(4n-2n_j)}\left\{\frac{6n}{(3n-2n_j)}\left\{\frac{3a^2}{4a_j^2}b_{3,2}-\frac{a}{2a_j}b_{3,3}-\frac{a^2}{4a_j^2}b_{3,4}\right\}\right.$$

$$\left.-\frac{3a^2}{a_j^2}b_{3,2}+\frac{3a}{a_j}b_{3,3}-\frac{a^2}{2a_j^2}b_{3,4}\right\}e_j\cos(3nt-2n_jt+3\varepsilon-2\varepsilon_j-\varpi_j) \qquad [18]$$

$$-\frac{m_j}{\mu}\frac{n^2}{(3n-2n_j)(5n-3n_j)}\left\{\frac{8n}{(4n-3n_j)}\left\{\frac{3a^2}{4a_j^2}b_{3,3}-\frac{a}{2a_j}b_{3,4}-\frac{a^2}{4a_j^2}b_{3,5}\right\}\right.$$

$$\left.-\frac{15}{4}\frac{a^2}{a_j^2}b_{3,3}+\frac{4a}{a_j}b_{3,4}-\frac{3a^2}{4a_j^2}b_{3,5}\right\}e_j\cos(4nt-3n_jt+4\varepsilon-3\varepsilon_j-\varpi_j) \qquad [19]$$

$$+\frac{m_j}{\mu}\frac{n^2}{2n_j(2n-2n_j)}\left\{\frac{2n}{(n-2n_j)}\left\{\frac{2a^2}{a_j^2}-\frac{a^2}{2a_j^2}b_{3,0}-\frac{a}{2a_j}b_{3,1}+\frac{3a^2}{4a_j^2}b_{3,2}\right\}\right.$$

$$\left.+\frac{2a^2}{a_j^2}+\frac{a^2}{a_j^2}b_{3,0}-\frac{a}{a_j}b_{3,1}\right\}e_j\cos(nt-2n_jt+\varepsilon-\varepsilon_j+\varpi_j) \qquad [20]$$

$$-\frac{m_j}{\mu}\frac{n^2}{(n-3n_j)(3n-3n_j)}\left\{\frac{4n}{(2n-3n_j)}\left\{-\frac{a^2}{4a_j^2}b_{3,1}-\frac{a}{2a_j}b_{3,2}+\frac{3a^2}{4a_j^2}b_{3,3}\right\}\right.$$

$$\left.+\frac{3a^2}{4a_j^2}b_{3,1}-\frac{2a}{a_j}b_{3,2}+\frac{3a^2}{4a_j^2}b_{3,3}\right\}e_j\cos(2nt-3n_jt+2\varepsilon-3\varepsilon_j+\varpi_j) \qquad [21]$$

$$-\frac{m_j}{\mu}\frac{n^2}{(2n-4n_j)(4n-4n_j)}\left\{\frac{6n}{(3n-4n_j)}\left\{-\frac{a^2}{4a_j^2}b_{3,2}-\frac{a}{2a_j}b_{3,3}+\frac{3a^2}{4a_j^2}b_{3,4}\right\}\right.$$

$$\left.+\frac{a^2}{a_j^2}b_{3,2}-\frac{3a}{a_j}b_{3,3}+\frac{3a^2}{2a_j^2}b_{3,4}\right\}e_j\cos(3nt-4n_jt+3\varepsilon-4\varepsilon_j+\varpi_j) \qquad [22]$$

$$-\frac{m_j}{\mu}\frac{n^2}{(3n-5n_j)(5n-5n_j)}\left\{\frac{8n}{(4n-5n_j)}\left\{-\frac{a^2}{4a_j^2}b_{3,3}-\frac{a}{2a_j}b_{3,4}+\frac{3a^2}{4a_j^2}b_{3,5}\right\}\right.$$

$$\left.+\frac{5}{4}\frac{a^2}{a_j^2}b_{3,3}-\frac{4a}{a_j}b_{3,4}+\frac{9}{4}\frac{a^2}{a_j^2}b_{3,5}\right\}e_j\cos(4nt-5n_jt+4\varepsilon-5\varepsilon_j+\varpi_j) \qquad [23]$$

Substituting in the equations of p. 45, for R_n their values and neglecting e^2, ee_j, e_j^2, and $\sin^2\frac{i_j}{2}$,

Expression for the longitude when $a < a_1$.

$$\lambda = n\,t + \varepsilon + e\sin(n\,t + \varepsilon - \varpi)$$

$$+\frac{n}{(n-n_1)}\left\{2\,r_1 - \frac{m_1\,n}{\mu\,(n-n_1)}\left(\frac{a^2}{a_1^2} - \frac{a}{a_1}\,b_{1,1}\right)\right\}\sin\left(n\,t - n_1 t + \varepsilon - \varepsilon_1\right) \quad [1]$$

$$+\frac{n}{(2\,n-2\,n_1)}\left\{2\,r_2 + \frac{m_1\,n}{\mu\,(n-n_1)}\,\frac{a}{a_1}\,b_{1,2}\right\}\sin\left(2\,n\,t - 2\,n_1 t + 2\,\varepsilon - 2\,\varepsilon_1\right) \quad [2]$$

$$+\frac{n}{(3\,n-3\,n_1)}\left\{2\,r_3 + \frac{m_1\,n\,a}{\mu\,(n-n_1)\,a_1}\,b_{1,3}\right\}\sin\left(3\,n\,t - 3\,n_1 t + 3\,\varepsilon - 3\,\varepsilon_1\right) \quad [3]$$

$$+\frac{n}{(4\,n-4\,n_1)}\left\{2\,r_4 + \frac{m_1\,n\,a}{\mu\,(n-n_1)\,a_1}\,b_{1,4}\right\}\sin\left(4\,n\,t - 4\,n_1 t + 4\,\varepsilon - 4\,\varepsilon_1\right) \quad [4]$$

$$+\frac{n}{(5\,n-5\,n_1)}\left\{2\,r_5 + \frac{m_1\,n\,a}{\mu\,(n-n_1)\,a_1}\,b_{1,5}\right\}\sin\left(5\,n\,t - 5\,n_1 t + 5\,\varepsilon - 5\,\varepsilon_1\right) \quad [5]$$

$$+\frac{n}{n_1}\left\{2\left(r_6 + \frac{r_1}{2}\right) + \frac{m_1\,n}{\mu\,n_1}\left(-\frac{3\,a^2}{2\,a_1^2} + \frac{3\,a^2}{2\,a_1^2}\,b_{3,0} - \frac{a^2}{2\,a_1^2}\,b_{3,1} - \frac{a^2}{4\,a_1^2}\,b_{3,2}\right)\right.$$
$$\left. - \frac{m_1\,n}{\mu\,(n-n_1)}\left(\frac{a^2}{a_1^2} - \frac{a}{a_1}\,b_{1,1}\right)\right\}e\sin\left(n_1 t + \varepsilon - \varpi\right) \quad [6]$$

$$+\frac{n}{(2\,n-n_1)}\left\{2\left(r_8 + \frac{r_1}{2}\right) - \frac{m_1\,n}{\mu\,(2\,n-n_1)}\left(\frac{a^2}{2\,a_1^2} - \frac{a^2}{2\,a_1^2}\,b_{3,0} - \frac{a^3}{2\,a_1^3}\,b_{3,1} + \frac{3\,a^2}{4\,a_1^2}\,b_{3,2}\right)\right.$$
$$\left. - \frac{m_1\,n}{\mu\,(n-n_1)}\left(\frac{a^2}{a_1^2} - \frac{a}{a_1}\,b_{1,1}\right)\right\}e\sin\left(2\,n\,t - n_1 t + 2\,\varepsilon - \varepsilon_1 - \varpi\right) \quad [8]$$

$$+\frac{n}{(3\,n-2\,n_1)}\left\{2\left(r_9 + \frac{r_2}{2}\right) - \frac{2\,m_1\,n}{\mu\,(3\,n-2\,n_1)}\left(-\frac{a^2}{4\,a_1^2}\,b_{3,1} - \frac{a^3}{2\,a_1^3}\,b_{3,2} + \frac{3\,a^2}{4\,a_1^2}\,b_{3,3}\right)\right.$$
$$\left. + \frac{m_1\,n\,a}{\mu\,(n-n_1)\,a_1}\,b_{1,2}\right\}e\sin\left(3\,n\,t - 2\,n_1 t + 3\,\varepsilon - 2\,\varepsilon_1 - \varpi\right) \quad [9]$$

$$+\frac{n}{(4\,n-3\,n_1)}\left\{2\left(r_{10} + \frac{r_3}{2}\right) - \frac{3\,m_1\,n}{\mu\,(4\,n-3\,n_1)}\left(-\frac{a^2}{4\,a_1^2}\,b_{3,2} - \frac{a^3}{2\,a_1^3}\,b_{3,3} + \frac{3\,a^2}{4\,a_1^2}\,b_{3,4}\right)\right.$$
$$\left. + \frac{m_1\,n\,a}{\mu\,(n-n_1)\,a_1}\,b_{1,3}\right\}e\sin\left(4\,n\,t - 3\,n_1 t + 4\,\varepsilon - 3\,\varepsilon_1 - \varpi\right) \quad [10]$$

$$+\frac{n}{(5\,n-4\,n_1)}\left\{2\left(r_{11} + \frac{r_4}{2}\right) - \frac{4\,m_1\,n}{\mu\,(5\,n-4\,n_1)}\left(-\frac{a^2}{4\,a_1^2}\,b_{3,3} - \frac{a^3}{2\,a_1^3}\,b_{3,4} + \frac{3\,a^2}{4\,a_1^2}\,b_{3,5}\right)\right.$$
$$\left. + \frac{m_1\,n\,a}{\mu\,(n-n_1)\,a_1}\,b_{1,4}\right\}e\sin\left(5\,n\,t - 4\,n_1 t + 5\,\varepsilon - 4\,\varepsilon_1 - \varpi\right) \quad [11]$$

$$+\frac{n}{(n-2\,n_1)}\left\{2\left(r_{12} + \frac{r_2}{2}\right) - \frac{2\,m_1\,n}{\mu\,(n-2\,n_1)}\left(\frac{3\,a^2}{4\,a_1^2}\,b_{3,1} - \frac{a^3}{2\,a_1^3}\,b_{3,2} - \frac{a^2}{4\,a_1^2}\,b_{3,3}\right)\right.$$
$$\left. + \frac{m_1\,n\,a}{\mu\,(n-n_1)\,a_1}\,b_{1,2}\right\}e\sin\left(n\,t - 2\,n_1 t + \varepsilon - 2\,\varepsilon_1 + \varpi\right) \quad [12]$$

Expression for the longitude when $a < a_i$.

$$+\frac{n}{(3n-3n_i)}\left\{2\left(r_{13}+\frac{r_3}{2}\right)-\frac{3m_in}{\mu(2n-3n_i)}\left(\frac{3a^2}{4a_i^2}b_{3,2}-\frac{a^3}{2a_i^3}b_{3,3}-\frac{a^3}{4a_i^3}b_{3,4}\right)\right.$$
$$\left.+\frac{m_ina}{\mu(n-n_i)a_i}b_{1,3}\right\}e\sin(2nt-3n_it+2\varepsilon-3\varepsilon_i+\varpi) \quad [13]$$

$$+\frac{n}{(3n-4n_i)}\left\{2\left(r_{14}+\frac{r_4}{2}\right)-\frac{4m_in}{\mu(3n-4n_i)}\left(\frac{3a^2}{4a_i^2}b_{3,3}-\frac{a^3}{2a_i^3}b_{3,4}-\frac{a^3}{4a_i^3}b_{3,5}\right)\right.$$
$$\left.+\frac{m_ina}{\mu(n-n_i)a_i}b_{1,4}\right\}e\sin(3nt-4n_it+3\varepsilon-4\varepsilon_i+\varpi) \quad [14]$$

$$+\frac{2n}{n_i}r_{15}e_i\sin(n_it+\varepsilon_i-\varpi_i) \quad [15]$$

$$+\frac{n}{(2n-n_i)}\left\{2r_{17}-\frac{2m_in}{\mu(2n-n_i)}\left(\frac{3a^2}{4a_i^2}b_{3,1}-\frac{a}{2a_i}b_{3,2}\right.\right.$$
$$\left.\left.-\frac{a^2}{4a_i^2}b_{3,3}\right)\right\}e_i\sin(2nt-n_it+2\varepsilon-\varepsilon_i-\varpi_i) \quad [17]$$

$$+\frac{n}{(3n-2n_i)}\left\{2r_{18}-\frac{3m_in}{\mu(3n-2n_i)}\left(\frac{3a^2}{4a_i^2}b_{3,2}-\frac{a}{2a_i}b_{3,3}\right.\right.$$
$$\left.\left.-\frac{a^2}{4a_i^2}b_{3,4}\right)\right\}e_i\sin(3nt-2n_it+3\varepsilon-2\varepsilon_i-\varpi_i) \quad [18]$$

$$+\frac{n}{(4n-3n_i)}\left\{2r_{19}-\frac{4m_in}{\mu(4n-3n_i)}\left(\frac{3a^2}{4a_i^2}b_{3,3}-\frac{a}{2a_i}b_{3,4}\right.\right.$$
$$\left.\left.-\frac{a^2}{4a_i^2}b_{3,5}\right)\right\}e_i\sin(4nt-3n_it+4\varepsilon-3\varepsilon_i-\varpi_i \quad [19]$$

$$+\frac{n}{(n-2n_i)}\left\{2r_{20}-\frac{m_in}{\mu(n-2n_i)}\left(\frac{2a^2}{a_i^2}-\frac{a^2}{2a_i^2}b_{3,0}-\frac{a}{2a_i}b_{3,1}\right.\right.$$
$$\left.\left.+\frac{3}{4}\frac{a^2}{a_i^2}b_{3,2}\right)\right\}e_i\sin(nt-2n_it+\varepsilon-2\varepsilon_i+\varpi_i) \quad [20]$$

$$+\frac{n}{(2n-3n_i)}\left\{2r_{21}-\frac{m_in}{\mu(2n-3n_i)}\left(-\frac{a^2}{4a_i^2}b_{3,1}-\frac{a}{2a_i}b_{3,2}\right.\right.$$
$$\left.\left.+\frac{3a^2}{4a_i^2}b_{3,3}\right)\right\}e_i\sin(2nt-3n_it+2\varepsilon-3\varepsilon_i+\varpi_i) \quad [21]$$

$$+\frac{n}{(3n-4n_i)}\left\{2r_{22}-\frac{m_in}{\mu(3n-4n_i)}\left(-\frac{a^2}{4a_i^2}b_{3,2}-\frac{a}{2a_i}b_{3,3}\right.\right.$$
$$\left.\left.+\frac{3a^2}{4a_i^2}b_{3,4}\right)\right\}e_i\sin(3nt-4n_it+3\varepsilon-4\varepsilon_i+\varpi_i) \quad [22]$$

The expression for the tangent of the latitude has already been given, p. 49.

When the latitude is reckoned from the plane of the orbit of the planet P, the following terms must be added, in the general case where ι is not equal to zero, to that expression;

$$-\frac{n^2}{2(n-n_i)(3n-n_i)}\frac{a^2}{a_i^2}\tan\iota\, b_{3,1}\sin(2nt-n_it+2\varepsilon-\varepsilon_i-\nu)$$

$$+\frac{n^2}{2(n-n_i)(n+n_i)}\frac{a^2}{a_i^2}\tan\iota\, b_{3,1}\sin(n_it+\varepsilon_i-\nu_i)$$

$$-\frac{n^2}{2(2n-2n_i)(4n-2n_i)}\frac{a^2}{a_i^2}\tan\iota\, b_{3,2}\sin(3nt-2n_it+3\varepsilon-2\varepsilon_i-\nu)$$

$$-\frac{n^2}{4n_i(2n-2n_i)}\frac{a^2}{a_i^2}\tan\iota\, b_{3,2}\sin(nt-2n_it+\varepsilon-2\varepsilon_i+\nu)$$

$$-\frac{n^2}{2(3n-3n_i)(5n-3n_i)}\frac{a^2}{a_i^2}\tan\iota\, b_{3,3}\sin(4nt-3n_it+4\varepsilon-3\varepsilon_i-\nu)$$

$$+\frac{n^2}{2(n-3n_i)(3n-3n_i)}\frac{a^2}{a_i^2}\tan\iota\, b_{3,3}\sin(2nt-3n_it+2\varepsilon-3\varepsilon_i+\nu)$$

All the equations hitherto given apply to the case of an inferior disturbed by a superior planet, or when $a_i > a$, in order to render them applicable to the case when $a_i < a$ it is necessary to write a instead of a_i in the denominator of the terms multiplied by $b_{1,n}$, and a^3 instead of a_i^3 in the denominator of the terms multiplied by $b_{3,n}$, in the disturbing function R, but the expressions for the quantities q are not the same in this case.

It will I think be admitted that the expressions which occur in the theory of the disturbances of the planets are more simple in terms of the quantities of which the general symbol is b, than in terms of the partial differential coefficients of the quantity called A in the notation of the Mécanique Céleste. The development of the disturbing function R in terms of the differential coefficients $\frac{dA}{da}$, $\frac{dA}{da_i}$, &c. admits of reductions, so that it may be expressed in terms of the differential coefficients of A with respect to a only. In this state it has been left by Laplace as may be seen, vol. ii. p. 12, but the coefficients of the terms multiplied by the squares and products of the eccentricities may be expressed very simply in terms of the quantities of which the general symbol is b, by means of reductions, of which two exam-

ples are given in the Théor. Anal. vol. i. p. 362. Similar reductions are applicable to the terms in $r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)$ multiplied by the first power of the eccentricities.

In Laplace's notation

$$b_{1,0}=\tfrac{1}{2}b_{\frac{1}{2}}^{(0)},\quad b_{1,1}=b_{\frac{1}{2}}^{(1)},\quad b_{1,2}=b_{\frac{1}{2}}^{(2)},\quad b_{1,3}=b_{\frac{1}{2}}^{(3)}$$

$$b_{2,0}=\tfrac{1}{2}b_{\frac{3}{2}}^{(0)},\quad b_{2,1}=b_{\frac{3}{2}}^{(1)},\quad b_{2,2}=b_{\frac{3}{2}}^{(2)},\quad b_{2,3}=b_{\frac{3}{2}}^{(3)}$$

$$b_{3,0}=\tfrac{1}{2}b_{\frac{5}{2}}^{(0)},\quad b_{3,0}=b_{\frac{5}{2}}^{(1)},\quad b_{3,2}=b_{\frac{5}{2}}^{(2)},\quad b_{3,3}=b_{\frac{5}{2}}^{(3)}$$

$$\frac{a}{a_{,}}b_{2,0}-\tfrac{1}{2}b_{2,1}=-\tfrac{1}{2}\frac{\mathrm{d}b_{\frac{3}{2}}^{(0)}}{\mathrm{d}\alpha},\qquad \frac{a}{a_{,}}b_{2,1}-b_{2,0}-\tfrac{1}{2}b_{2,2}=-\frac{\mathrm{d}b_{\frac{3}{2}}^{(1)}}{\mathrm{d}\alpha}$$

$$\frac{a}{a_{,}}b_{2,2}-\tfrac{1}{2}b_{2,1}-\tfrac{1}{2}b_{2,3}=-\frac{\mathrm{d}b_{\frac{3}{2}}^{(2)}}{\mathrm{d}\alpha},\qquad \frac{a}{a_{,}}b_{2,3}-\tfrac{1}{2}b_{2,2}-\tfrac{1}{2}b_{2,4}=-\frac{\mathrm{d}b_{\frac{3}{2}}^{(3)}}{\mathrm{d}\alpha}$$

$$3\left\{\frac{a}{a_{,}}b_{3,0}-\tfrac{1}{2}b_{3,1}\right\}=-\tfrac{1}{2}\frac{\mathrm{d}b_{\frac{5}{2}}^{(0)}}{\mathrm{d}\alpha},\qquad 3\left\{\frac{a}{a_{,}}b_{3,1}-b_{3,0}-\tfrac{1}{2}b_{3,2}\right\}=-\frac{\mathrm{d}.b_{\frac{5}{2}}^{(1)}}{\mathrm{d}\alpha}$$

$$3\left\{\frac{a}{a_{,}}b_{3,2}-\tfrac{1}{2}b_{3,1}-\tfrac{1}{2}b_{3,3}\right\}=-\frac{\mathrm{d}.b_{\frac{5}{2}}^{(2)}}{\mathrm{d}\alpha},\qquad 3\left\{\frac{a}{a_{,}}b_{3,3}-\tfrac{1}{2}b_{3,2}-\tfrac{1}{2}b_{3,4}\right\}=-\frac{\mathrm{d}.b_{\frac{5}{2}}^{(3)}}{\mathrm{d}\alpha}$$

The numerical values of these quantities are given for the principal planets in the third volume of the Mécanique Céleste.

The following numerical examples will serve to explain the expressions given above, and to show their accuracy, the results agreeing exactly with those given in the Mécanique Céleste.

$$\frac{r}{a}=1+\frac{m_{,}}{\mu}\left\{\frac{a^{3}}{a_{,}^{3}}b_{3,0}-\frac{a^{3}}{2a_{,}^{3}}b_{3,1}\right\}-\mathrm{e}\left\{1+\frac{m_{,}}{\mu}\left\{\frac{*\,a^{3}}{3a_{,}^{3}}b_{3,0}-\frac{5}{12}\frac{a^{3}}{a_{,}^{3}}b_{3,1}\right\}\right\}\cos\left(n\,t+\varepsilon-\varpi\right)$$

$$-r_{,}\cos\left(n\,t-n_{,}t+\varepsilon-\varepsilon_{,}\right)$$ See p. 55.

* The coefficient of cos $(n\,t+\varepsilon-\varpi)$ p. 52 line 11, and p. 55 line 3, should be

$$-\mathrm{e}\left\{1+\frac{m_{,}}{\mu}\left\{\frac{a^{3}}{3a_{,}^{3}}b_{3,0}-\frac{5}{12}\frac{a^{3}}{a_{,}^{3}}b_{3,1}\right\}\right\}.$$

$$r_1 = \frac{m_1}{\mu}\frac{n^2}{(2n-n_1)n_1}\left\{\frac{2n}{(n-n_1)}\left(\frac{a^2}{a_1^2}-\frac{a}{a_1}b_{1,1}\right)+\frac{a^2}{a_1^2}+\frac{a^2}{a_1^2}\left(\frac{a}{a_1}b_{2,1}-b_{2,0}-\frac{1}{2}b_{2,2}\right)\right\}$$ See p. 57.

$$\lambda = nt+\varepsilon+\frac{n}{(n-n_1)}\left\{2r_1-\frac{m_1 n}{\mu(n-n_1)}\left(\frac{a^2}{a_1^2}-\frac{a}{a_1}b_{1,1}\right)\right\}\sin(nt-n_1t+\varepsilon-\varepsilon_1)$$

In the theory of Jupiter disturbed by Saturn,

$$\frac{a}{a_1} = \cdot 54531725, \qquad b_{1,1} = \cdot 6206406, \qquad b_{2,0} = \frac{4\cdot 358387}{2} = 2\cdot 179193$$

$$b_{2,1} = 3\cdot 185493 \qquad b_{2,2} = 2\cdot 082131 \qquad \frac{m_1}{\mu} = \frac{1}{3359\cdot 4}$$

$$n = 337210\cdot 78 \qquad n_1 = 135792\cdot 34$$

$$a = 5\cdot 20116636 \qquad e = \cdot 0480767$$

See Méc. Cél. vol. iii. p. 61 & 82.

Whence

$$\frac{a^3}{a_1^3}b_{2,0} = \cdot 353381, \qquad \frac{a^3}{2a_1^2}b_{2,1} = \cdot 479636 \qquad \frac{a^3}{a_1^3}b_{2,0}-\frac{a^3}{2a_1^2}b_{2,1} = -\cdot 120255$$

$$\log.\ \cdot 120255 = 9\cdot 0801033$$

$$\log.\ 5\cdot 20116636 = 0\cdot 7161007$$

$$9\cdot 7962040$$

$$\log.\frac{3\mu}{m_1} = 4\cdot 0033829$$

$$5\cdot 7928211 = \log.\ \cdot 0000620613 \text{ minus}$$

Laplace has . . . . . ·0000620566 minus

See Méc. Cél. vol. iii. p. 121, line 5.

$$\log.\left\{\frac{5}{12}\frac{a^2}{a_1^2}b_{2,1}-\frac{a^3}{3a_1^3}b_{2,0}\right\} = 9\cdot 4423277$$

$$\log. e = 8\cdot 6819347$$

$$\log. a = 0\cdot 7161007$$

$$8\cdot 8403631$$

$$\log.\frac{\mu}{m_1} = 3\cdot 5262617$$

$$5\cdot 3141014 = \log.\ \cdot 0000206111 +$$

Laplace has . . . . . ·0000206111 the sign omitted,

Méc. Cél. vol. iii. p. 122, line 28.

Calculation of r_1, see p. 57.

$\log. n_1 = 5{\cdot}1328751$

$\log. n = 5{\cdot}5279013$

$9{\cdot}6049738 = \log. {\cdot}4026928$

$1 - \frac{n_1}{n} = {\cdot}5973072$

$\frac{a}{a_1} b_{3,1} - b_{3,0} - \frac{1}{2} b_{3,2} = -1{\cdot}48315$

$\log. 1{\cdot}48315 = {\cdot}1711851$

$\log. \frac{a^4}{a_1^3} = 9{\cdot}4732982$

$9{\cdot}6444833 = \log. {\cdot}441045$

$\log. 1{\cdot}5973072 = {\cdot}2033884$

$\log. {\cdot}4026928 = 9{\cdot}6049738$

$\log. \ 3359{\cdot}4 = 3{\cdot}5262617$

$3{\cdot}3346239$

$r_1 = -{\cdot}0001301383$

similarly $r_2 = {\cdot}000556924$, $r_3 = {\cdot}00005809$, $r_4 = {\cdot}000015045$, $r_5 = {\cdot}0000049785$

$\frac{a^2}{a_1^2} - \frac{a}{a_1} b_{1,1} = -{\cdot}041075$

$\log. {\cdot}041075 = 8{\cdot}6135776$

$\log. \frac{n - n_1}{n} = 0{\cdot}7761940$

$8{\cdot}8373836 = \log. {\cdot}0687676$

2

$-{\cdot}1375352$

$-{\cdot}441045$

$-{\cdot}5785802$

$\frac{a^2}{a_1^2} = {\cdot}297371$

$-{\cdot}281209$

$\log. {\cdot}281209 = 9{\cdot}4490293$

$3{\cdot}3346259$

$6{\cdot}1144054 = \log. {\cdot}0001301383$

$\log. a = {\cdot}7161007$

$6{\cdot}8305061 = \log. {\cdot}000676871 +$

Laplace has . . . . ${\cdot}000676876 +$ p. 120.

Calculation of the coefficient of $\sin(nt - n_1 t + \varepsilon - \varepsilon_1)$ in the value of λ. See p. 59.

$2 r_1 = -{\cdot}0002602766$

$\quad {\cdot}0000204702$

$\log. \ {\cdot}0002398064 = 6{\cdot}3798601$

$\log. \frac{n - n_1}{n} = 9{\cdot}7761940$

$6{\cdot}6036661$

$5{\cdot}8038801$ see p. 49.

$2{\cdot}4075461 = \log. 255{\cdot}591$ minus.

$\log. \frac{n}{n - n_1} \left\{ \frac{a^2}{a_1^2} - \frac{a}{a_1} b_{1,1} \right\} = 8{\cdot}8373836$

$\log. 3359{\cdot}4 = 3{\cdot}5262617$

$5{\cdot}3111219 = \log. {\cdot}0000204702$

Laplace has . . . 255·5917 the sign omitted, p. 120.

In the notation of the Méc. Cél. vol. iii. p. 120.

$n = n^{iv}$, $\quad n_1 = n^{v}$.

The reader is requested to make the following corrections.

The expression for r^{-3} Phil. Trans. 1830, p. 345, should be

$$r^{-3} = a^{-3}\left\{1 + \frac{3}{2}e^2\left(1 + \frac{5}{4}e^2\right) + 3e\left(1 + \frac{9}{8}e^2\right)\cos(nt - \varpi)\right.$$
$$+ \frac{9}{2}e^2\left(1 + \frac{7}{9}e^2\right)\cos(2nt - 2\varpi) + \frac{53}{8}e^3\cos(3nt - 3\varpi)$$
$$\left. + \frac{77}{8}e^4\cos(4nt - 4\varpi)\right.$$

The third term (multiplied by $\mathrm{d}t$) in the equations of p. 23, line 4 and 6, and the second term in the equation of the same page, line 11, must be suppressed.

p. 30, line 21, read $+ m_i\left\{\frac{a}{a_i^2}\left(\cos^2\frac{I_i}{2} - \frac{e^2 + e_i^2}{2}\right)\right.$ for $m_i\left\{-\frac{a}{a_i^2}\left(\cos^2\frac{I_i}{2} - \frac{e^2 + e_i^2}{2}\right)\right.$

p. 32, line 5, read $+ m_i\left\{\frac{2a}{a_i^2} - \frac{a}{2a_i^2}b_{3,0} - \frac{a_i^2}{2a^2}b_{3,1}\right.$ &c. for $+ m_i\left\{\frac{a}{2a_i^2} - \frac{a}{2a_i^2}b_{3,0} - \frac{a_i^2}{2a_i^3}b_{3,1}\right.$ &c.

p. 39, line 9, read $-\frac{21}{8}\frac{a^2}{a_i^2}ee_i\cos(2t - z + x)$ for $-\frac{27}{8}\frac{a^2}{a_i^2}ee_i\cos(2t - z + x)$

p. 41, line 4, read $\frac{n_i^2}{n^2}$ for $\frac{n^2}{n_i^2}$.

IV. *On the Transient Magnetic State of which various Substances are susceptible.* *By* WILLIAM SNOW HARRIS, *Esq.* *Communicated by* DAVIES GILBERT, *Esq. V.P.R.S.*

Read June 17, 1830.

1. THE influence of bodies, not permanently magnetic, on the compass needle, has led to some further researches in magnetism of singular importance. The valuable papers of M. ARAGO and other eminent philosophers on this subject, communicated to the Royal Academy of Sciences, together with those of our no less talented countrymen which have appeared in the Royal Society's Transactions, are calculated to excite a deep interest in physics. In presenting to the Royal Society, after such valuable contributions to science, an account of some inquiries which I have myself been led to make in this branch of natural knowledge, I am encouraged by the belief, that whilst the deductions are for the most part derived from simple and direct experiments, the investigation has been carried on under new conditions; and that although some facts already made known, have been again referred to, it will nevertheless be found to contain results of sufficient consequence to render it not unworthy of notice.

I have been particularly led to persevere in this investigation, at various times, from having observed (Edinburgh Journal of Science, vol. v. p. 325.) that the reciprocal action of a rotating metallic disc, and a magnetic bar, is not considered by M. ARAGO as the result of any magnetism induced in the disc, but is attributed by him to some new force as yet undiscovered:—and also, from the circumstance, that the magnetic susceptibility of non-metallic bodies, seems not to have been in many instances satisfactorily established:—two inquiries of considerable importance.

2. When these researches were first commenced*, a curious effect was observed, evidently resulting from vibration, which it may be of consequence to mention.

A thin circular ring of copper of about half an inch wide, ten inches in

* In 1826.

diameter, and about $\frac{1}{10}$th of an inch thick, being delicately balanced on a fine point, by means of a transverse bar of wood carrying an agate centre, was placed on a firm screen of paper strained tight on a wood frame, and covered by a glass shade; two powerful magnetic bars were caused to revolve in a horizontal plane, beneath the screen, with their poles immediately under the ring. The ring, as in all the previous experiments of the same kind, soon acquired motion in the direction of the revolving bars, which could be arrested and reversed at pleasure, by changing the direction of the rotation. It was however soon discovered, that a similar effect could be produced when the magnets were not present, merely by the action of the rotating apparatus. This last consisted of a train of wheels resting on a firm pavement of thick stone; the frame carrying the screen being supported over it on blocks of oak.

As it was of consequence to ascertain how far small vibratory impulses communicated to the screen, could cause the phenomena just observed, a continuous but gentle vibration was induced from one corner of it, by a slight tapping, which, after a few minutes, communicated motion to the ring in a constant direction; on transferring the vibrations to the opposite side of the frame, the ring was again brought to rest, and caused to revolve in an opposite direction. A similar result was obtained when a compass needle, and light needles of other substances were substituted for the ring of copper.

3. These results led me to place the copper circle in an exhausted receiver, on a strong plate of glass, and to suspend the whole on a convenient frame by means of lines passing from firm walls; the revolving bars being placed as before, immediately under it. In this instance, however, I failed at the same distance to move the ring, nor could a fine compass needle similarly placed be made to deviate more than a few degrees from its meridian, when exposed to the influence of a rapidly revolving disc. This result is quite sufficient to show how minute a cause may render delicate investigations with a rotating body unsatisfactory; more especially when connected with a heavy apparatus: even in the case of screens it does not seem quite clear that certain impulses may not be propagated through them, of sufficient magnitude to act on a very finely suspended needle or disc; when the screens are thin and porous, the chance of this is greatly increased, and in cases where a screen only is employed without a shade; similar results would doubtless ensue from vortices

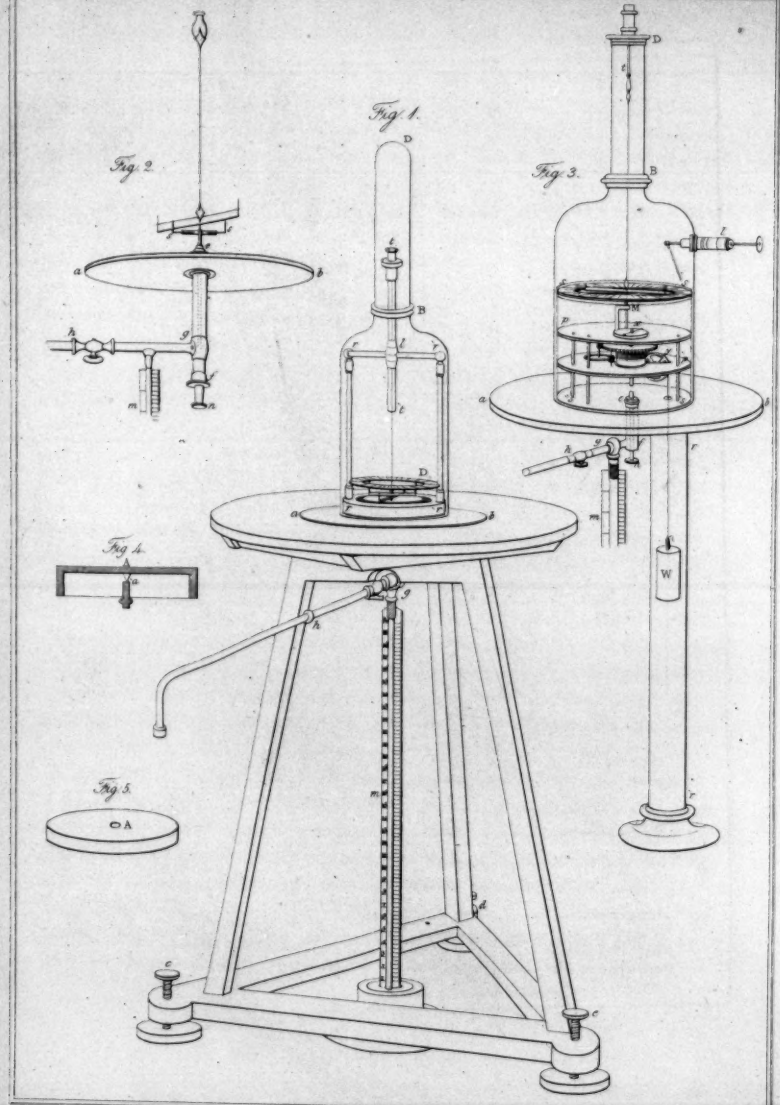


Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
A
W

Fig. 8.

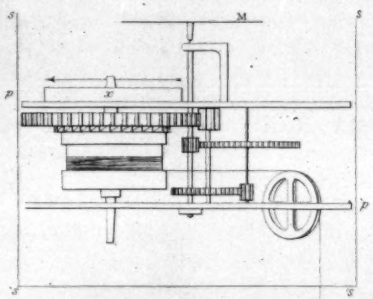


Fig. 6.

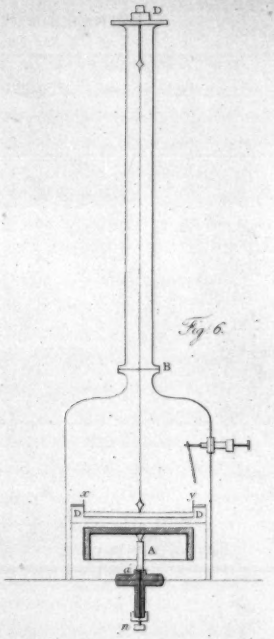


Fig. 9.

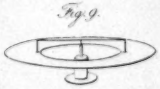


Fig. 7.

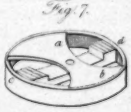


Fig. 10.

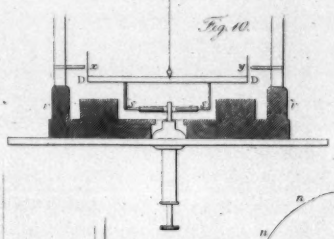


Fig. 11.

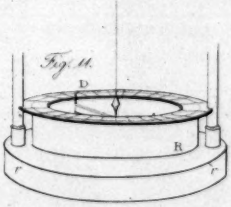
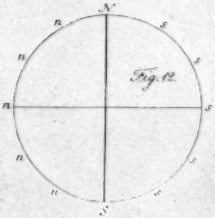


Fig. 12.



in the whole mass of the surrounding air. I do not however mean to infer that such sources of fallacy have not been fully calculated on, or guarded against in the many valuable papers already alluded to: I rather mention them as a reason for adopting a method of experimenting, differing from the preceding, and which might insure an unmixed result. To this end the experiments were continued in an exhausted receiver, effectually screened from vortices, and liberated from the influence of a resisting medium; and in case of employing a rotating body, with a rotation continued so smoothly and evenly, as not to cause any vibration capable of inducing motion.

4. The following is a description of the different mechanical arrangements resorted to. A circular plate of close fine-grained slate, *a b*, fig. 1, 2, 3, of about half an inch thick and a foot in diameter, being carefully ground true, in order to fit it for the purposes of an air-pump plate, was nicely finished by rubbing it over with a mixture of oil and wax.* This plate is supported on a firm stand and frame, as represented in fig. 1, furnished with levelling screws *c*, *d*, *e*. A short brass cylinder *e*, *f*, fig. 2 & 3, is fixed under the plate by means of a hole in the centre, and two brass collars ground true to its surface, and between which the plate is compressed: so that with a little grease, and a strong nut at *e*, fig. 2, 3, the joint is rendered air-tight. The cylinder *e f* connects laterally with a short pipe and stopcock *g h* leading to a good air-pump; a long barometer gauge *m*, fig. 1, 2 & 3, being attached to the pipe to indicate the state of the exhaustion. A straight rod of brass *n e*, having a milled head at *n*, passes through a compressed collar of fine cork at *f*, and is sufficiently free above in the cylinder to allow of the action of the pump; this rod is occasionally employed when furnished with a transverse bar at *e*, having two stops *s*, *s*, fig. 2 & 10, to liberate a magnetic bar in an exhausted medium, when retained at any angle of deviation from its meridian, or to effect any other required operation, as in fig. 3.

5. The revolving body A, fig. 4 & 5, consists of a circular disc of about five inches diameter, and 0.125 of an inch thick; united to a deep ring of the same

* The application of this substance to the purposes of an air pump-plate will be found extremely convenient, more especially when it is desirable to avoid the presence of a large mass of metal. It is a very economical and efficient substitute for glass or brass, and retains its form without liability to warp; and is an application as far as I can learn quite new.

substance, or otherwise to a ring of lead. This disc has smooth rounded edges, and is balanced on a fine central point A of hardened steel, in such way that the centre of gravity and centre of motion coincide as nearly as possible; the whole is sustained on a small agate cup, set in a short cylindrical piece of brass *a*, fig. 4, and screwed, when required, into the shoulder of the brass cylinder which passes through the round hole of the pump-plate as represented in fig. 6. The sliding rod *n e* above mentioned is in this case withdrawn for a short distance within the cylinder. The disc is set in motion at the rate required by means of a line wound rapidly off it, from a train of wheels; it is then covered with a screen, and with a receiver, as in fig. 6, and the exhaustion made as perfect as possible; in which case it will continue to revolve for a considerable time.

A simple or compound magnet is put into rapid rotation in a similar way, being previously mounted on a ring of lead; besides straight bars, a compound magnet was employed for this purpose, having its extremities turned up in a perpendicular direction for about an inch, as in *c d*, fig. 7, and occasionally a circular disc of magnetised steel, fig. 5. The former has its point of support fixed in a transverse bar of brass *a b*, fig. 7, which projects in a circular hole drilled through the centre of the bars.*

6. When it became necessary to measure with great precision the rapidity of rotation, the machine represented in fig. 3 & 8 was employed. It consists of a metallic disc M kept in motion by a delicate train of wheels and pinions *p p'*. The rapidity of the rotation being accurately measured by an index *x*, moving on a small circle divided into twenty parts. The weight W, fig. 3, which by its descent keeps up the rotation, falls in a tall narrow receiver *r r'*, supported against the under surface of the pump-plate.

A circular hole is drilled at *s'* through the plate for the silk cord to pass, from the pulley at *y*; by means of this hole also, the receiver *r r'* is exposed to the action of the pump. The machine is wound up and set in motion by means of the brass rod *n e*, fig. 3, already described (4), without disturbing the state of the exhaustion; the extremity of the rod being formed into a key.

The revolving disc is screened by glass or any other substance, over which

* This method of obtaining a rapid rotation in an exhausted receiver, suggested itself on perusing the account of Mr. Serson's horizontal speculum in the Royal Society's Transactions.

is placed a graduated circle of fine milled board, sustained in a light circular ring of wood. The screen and ring are moveable with friction in a hollow cylinder of glass *s s s' s''*, fig. 3, so as to be adjusted at any required altitude.

7. The receivers employed to inclose these various bodies are for the most part such as are represented in fig. 1, 3 & 6; they have an open neck at B ground to a flat surface which admits of the altitude being increased by the addition of a second narrow receiver B D: this last is also ground flat at the point of union; thus, by the aid of a little grease, an air-tight joint is easily obtained in the usual way. The upper receiver B D is either closed above, or open, with a ground flat rim in order to apply a brass plate D carrying a brass rod *t* in an air-tight collar. The rod *t* is for the convenience of raising or depressing light discs or needles suspended in the receiver below: these receivers vary from eight to forty inches in altitude, and from one to three inches in diameter; so that the filament of silk for suspension may, when required, be upwards of four feet in length.

8. The substances exposed to the influence of the revolving bodies are formed into light needles or bars, or otherwise into flat circular rings, fig. 9, and are either suspended or nicely balanced on points. For the purpose of retaining them at rest until the exhaustion is complete, and when the stop *e*, fig. 2, already explained (4) cannot be used, there is an angular lever *l*, fig. 3, which passes in an air-tight collar through the side of the receiver, carrying at its extremity a common reed *s*, which being tubular is easily fixed on it; hence it can be extended at pleasure.

9. The following are the results of some experiments with the apparatus above described.

(*a*) A circular disc of copper being put into rotation at the rate of 500 revolutions in a minute, was left unscreened under a receiver, and a fine reed suspended over it by a filament of silk; the reed was retained at rest, until the exhaustion was completed to within 0.4 of an inch of a good barometer, when it was set free. The reed rotated freely at any distance at which it could be placed from the revolving disc, and which at the greatest amounted to eight inches. The motion of the reed diminished with the rapidity of the rotation of the disc and with the torsion of the silk; but its motion could be again restored by admitting a small quantity of air into the receiver.

(*b*) In a tall receiver of about two feet high a reed similarly placed rotated at a distance of eighteen inches from the revolving disc, when the exhaustion was within six inches; but it remained at rest at a foot distance, when the exhaustion was carried to within 0.4 of an inch.

(*c*) The rotating disc being screened by a thin plate of glass or varnished paper fitted close upon a short hollow cylinder of glass or wood, the reed remained at rest, although placed within one fifth of an inch of the screen.

(*d*) A cylindrical magnetic bar revolved rapidly in an exhausted medium, when within one fifth of an inch distance from the disc; but only deviated from the meridian 40 or 50 degrees when the screen above mentioned was interposed. The disc in this experiment revolved at the rate of 400 revolutions in the minute.

(*e*) Needles of glass, wood, and metallic substances not permanently magnetic, remained at rest when the rotation exceeded 600 revolutions in a minute, whether suspended by filaments of silk, or otherwise placed on points and supported on the glass screen. When the rotation exceeded 1000 revolutions in a minute, and the exhaustion was inconsiderable, they were sometimes slightly moved.

(*f*) Rotating magnetic discs and bars induced motion in metallic discs freely suspended at one fifth of an inch distance, when their thickness was about four times as great as ordinary tinfoil, and when the rotation exceeded 500 revolutions in a minute; the rotating body being screened by any intervening substance except iron, which last, as already observed by Mr. Herschel and Mr. Babbage, completely intercepted the effect *. But light discs of wood and paper, and discs of paper covered with silver or gold leaf, remained at rest when within $\frac{1}{10}$ths of an inch distance from the rotating body.

(*g*) A disc of iron or tempered steel did not communicate motion to any

* Philosophical Transactions of the Royal Society for 1825.

Although a sheet or two of tinned iron completely intercepts the influence of a revolving magnet on metallic discs generally, it does not appear to do so when the disc acted on is also of iron. I found, however, that a screen of sheet iron of about one fourth of an inch thick materially diminished the influence even on an iron disc, so that the rotation of an iron disc by the influence of a revolving magnet is reduced nearly to that of a copper disc of the same dimensions when the iron screen is not present. This curious effect of screens may possibly lead to some further elucidation of the nature of magnetic influence.

substance not permanently magnetic; nor could motion be induced by the action of such substances on each other, however near they could be placed with an intervening screen; although the rapidity of the rotating body was caused to exceed 1000 revolutions in a minute.

(*h*) A needle of soft iron was not influenced by a copper disc revolving at the rate of 600 times in a minute; but rotated freely when surrounded by a helix transmitting an electro-magnetic current.

(*i*) A disc of tempered steel not magnetic, after simple contact with a powerful magnet acquired sufficient force to induce motion in metallic rings.

(*k*) A needle of tempered steel was not influenced by a rotating disc of soft iron at the distance of one fifth of an inch, and revolving at the rate of 600 times in a minute; but the needle revolved rapidly after simple contact with a magnet.

(*l*) Magnetic needles delicately mounted on an horizontal axis so as to admit of motion in a vertical plane, remained at rest when the axis was perpendicular to the radius of the revolving disc; but the needles appeared to be carried in the direction of its motion when the axis was turned in the same plane.

10. The above facts seem to show very clearly, that the presence of permanent magnetism, and a susceptibility of magnetic induction are essential conditions in the phenomena of rotation hitherto observed: at least this deduction is fair for all distances between the bodies at which they can be well placed with an intervening screen: without a screen it seems extremely difficult ever to arrive at a satisfactory result; since even in a very rare medium (Exp^{ts} (*a*), (*b*), (*c*), (*d*)) vortices are produced by the action of a rapidly revolving body, although with perfectly rounded and smooth edges, capable of carrying round light substances; and although separated from such substances by a considerable interval. So far therefore as this method of examining the influence which bodies not permanently magnetic can exert on each other extends, it seems not susceptible of the requisite precision; and is, therefore, in a great degree inadequate to detect any very minute and delicate force, which can be supposed to arise from such an influence.

11. The law according to which metallic or other discs in a state of rotation influence a permanent magnet seems to be directly as the rapidity of the rotation, and inversely as the squares of the distances between the attracting

bodies: at least for such distances as can be conveniently resorted to with an intervening screen. In order to investigate this, I resorted to the machine already described (6) and represented in fig. 3 & 8. By a few previous trials such weights were found as might in a short time impart to the rotating disc the respective velocities of 178.5, 357 and 714 revolutions in a minute; which numbers are to each other as the numbers 1, 2, 4: and by means of the index and graduated circle x, fig. 3 & 8, the deflections of a magnet could be taken when these velocities were attained. This point was determined by means of a valuable chronometer of a peculiar description which my friend Lieut.-Col. H. Smith, F.R.S. was so good as to lend me for the purpose; it can be set going, and stopped again at pleasure, and is capable of registering an observation to the $\frac{1}{10}$th part of a second. As the graduated circle is divided into twenty parts, it is presumed that the rate of motion can in this way be ascertained with sufficient accuracy; and thus any little acceleration caused by the descending weight is not of consequence.

12. A magnetic bar being suspended as in fig. 3, was in the first place accurately adjusted at the point of contact to the plane of the body intended to be put in motion, and which consisted of a flat ring of copper of an inch wide, and about 0.05 of an inch thick. The bar was then raised from the ring through a distance equal to five turns of a micrometer-screw at D, each turn of the screw being equal to the $\frac{1}{10}$th of an inch: the screen s was then interposed, and the whole covered by a receiver, and exhausted.

(n) The machine being set in motion, the deviation of the bar amounted to 24° when the velocity was 357 revolutions in a minute: on increasing the velocity to 714 revolutions in a minute, or double the former, it amounted to 56°. The exhaustion in these experiments was carried to within 0.5 of an inch. Taking the sines of the angles of deviation as a measure of the force urging the bar, we have the respective numbers .829038 and .406737, which are very nearly in the same ratio as the respective velocities; that is to say as 2 : 1.

(o) The bar being adjusted to within a distance of the ring equal to four turns of the screw, the deviation of the needle amounted to 38° when the velocity was 357 revolutions in a minute: on raising the bar by four additional turns of the screw, the deviation decreased to 9°. Taking the sines of these angles as before, we have for the corresponding distances, the numbers .615661

and .156434, which may be considered as very nearly in the inverse ratio of the squares of the respective distances.

In the following Table are given the deviations of the bar, corresponding to other distances and velocities.

TABLE I.

(A)			(B)			(C)		
Velocity of rotation, 178.5			Velocity of rotation, 357.			Velocity of rotation, 714.		
Distance in turns of Mic'-Screw.	Angle of Deviation.	Sine of Deviation.	Distance in turns of Mic'-Screw.	Angle of Deviation.	Sine of Deviation.	Distance in turns of Mic'-Screw.	Angle of Deviation.	Sine of Deviation.
4	° 18	.309017	4	° 38	.615661		°	
5	12	.207912	5	24	.406737	5	56	.829038
			6	16	.275637			
8	4.5	.078459	8	9	.156434			
10	3	.052336	10	6	.104528	10	12	.207912

It appears by the above Table, that the influence of the ring is directly as the rapidity of the motion, and inversely as the squares of the distances: we observe a little discrepancy in some of the numbers, but the general agreement is very close and remarkable. A complete agreement cannot be expected; for supposing even the most perfect manipulation, there will always arise in experiments of this kind many causes which disturb a numerical identity*.

13. Being desirous to extend these inquiries concerning the transient magnetic state, of which various substances appear to be susceptible, I subsequently laid aside the rotating discs, as a means of detecting these minute forces for the no less refined, and perhaps still less exceptionable method of a vibrating bar; since what is termed the magnetism of rotation, seems in fact to be

* In estimating the distance between the surfaces of the attracting bodies, the revolving disc was purposely made as thin as possible, so as to admit of its being considered without any sensible thickness. With respect to the magnet, the foregoing experiments with screens clearly show, that an intervening ferruginous mass completely intercepts the attractive force upon non-ferruginous substances. If any portion of the bar beneath the surface contiguous to the revolving body be supposed to operate upon the ring, such portion must necessarily act through the intervening iron; which it cannot do (Exp. *f*) (11). This view seems to derive much confirmation from experiment, since on vibrating an extremely thin needle of sheet steel over a metallic disc, the comparative results do not vary from those obtained by means of a bar of half an inch thick.

nothing more than the presence of a temporary magnetic development induced in successive points of a body by the action of a permanent magnet, the body being supposed in motion. If the body therefore be at rest, this development will restrain the motion of a bar vibrating near it, and so diminish the amplitude of its oscillations; and thus by determining the number of vibrations performed in a given arc, we may from thence arrive at a comparative value of the force in action.

To this effect the bar already mentioned (12) was suspended, and exposed to the influence of different substances, in the following manner.

Two perpendicular rods of glass $r\,r$, $r'\,r'$, fig. 1, furnished with foot- and cap-pieces are fixed in a solid block of mahogany $r\,r'$, fig. 1; these rods by the intervention of a short wood cylinder l, and two horizontal rods, also of glass, $r\,l$, $r'\,l$, sustain a glass tube $t\,t'$ which slides with friction through the cylinder. This tube is furnished with a cap-piece at t, through which passes a brass rod: the magnetic bar is suspended from this rod by a filament of silk, and is finally raised or depressed to the required altitude by a micrometer-screw at t: the altitude of the glass tube l being previously fixed. The centre of the mahogany block is hollowed into a cylindrical cavity as represented in fig. 10, and is firmly fixed upon the pump-plate by means of the screw and shoulder projecting through the brass collars (4). Its outer part is also depressed, leaving a cylindrical projection of about one fourth of an inch deep, 4.7 inches diameter, and 1.25 inch wide.

The graduated circle of stout card-board D already mentioned (6) slides with friction between the glass rods, so as to be easily adjusted at any given point. The bar is retained at the given angle from its meridian, and again set free, when the exhaustion is sufficiently complete, by means of the lever and double stop before explained (4), and which moves in the interior circular cavity of the block, fig. 10.

This method of arresting the bar seems to be of some consequence to the experiment; for if one pole only be checked, the force operating on the other, will for an obvious reason, give the bar a swinging motion, which is very undesirable, but which is here effectually prevented; so that when set free it will appear to oscillate as if mounted on a fixed centre. The checks by which the bar is thus arrested, are so contrived as to be independent of each other,

and are moveable with friction in a small tube of brass as at *ss*, figs. 2 and 10, and may consequently be adjusted with great nicety. The tube in which the checks are fixed is sustained in a horizontal direction in the extremity of the rod, which passes into the receiver through the centre of the pump-plate.

14. The substances to be submitted to experiment, are formed into rings as at R, fig. 11; in these the bar is caused to oscillate. Each ring is about one inch in height, of any convenient thickness, and 4.75 inches interior diameter, so as to admit of its being accurately adjusted on the cylindrical shoulder of the circular base, fig. 10.

15. In applying the method of an oscillating bar to the investigation of minute and transient magnetic forces, it seems essential to keep in view the following interesting fact; viz. the influence of bodies, not susceptible of permanent polarity, on the state of oscillation is such, that the amplitude of the vibrations only is diminished, not their duration: that is to say, whether a bar vibrate in free space, or otherwise near plates, or in rings of these substances, still the number of oscillations in a given time, considered as a unit of time, does not vary, although the bar is sooner brought to rest when under the influence of such bodies, than when freed from them, whatever substance be employed, and at whatever distance the influence be exerted.

This fact seems materially to distinguish the peculiar influence of non-magnetic substances, from a case of permanent polarity; by which last, an oscillating bar is not only brought to rest in less time, but the rate of vibration is very sensibly increased. We cannot therefore, as in the latter instance, resort to the common law of pendulums, and take the square of the number of oscillations performed in a given time as a measure of the force in action, since the time of each oscillation does not sensibly vary; we must therefore adopt some other method.

16. In order to arrive at a comparative value of the influence of any substance on a vibrating bar, I have been led to employ the following formula, $\left(\frac{a}{b} - 1\right) r$; which seems to apply in a very remarkable manner to the results of experiment; in which a represents the number of oscillations in a given arc in free space, b the number in the same arc, when exposed to the influence of a substance not permanently magnetic, and r the retarding force

by which the bar tends to rest in free space; the oscillations being supposed to take place in an exhausted receiver. Thus, if in free space 420 oscillations are performed before the arc of vibration is reduced from 45° to 10°, and the number of vibrations in the same arc, are 30 and 20 respectively, when the bar vibrates under the influence of two given substances taken in succession, and whose magnetic energies we propose to compare with each other; then the energy of the one may be expressed by $\left(\frac{420}{30} - 1\right) r$, and that of the other by $\left(\frac{420}{20} - 1\right) r$; that is to say, their energies will be to each other as 13 : 20 *.

17. Previously to examining the susceptibility of substances generally to magnetic influence, the number of vibrations was first determined between 45° and 10° in air, under a glass-shade; then the number in the same arc also in air, when the bar was surrounded by a ring of wood of one fifth of an inch thick, and an inch deep; each pole being $\frac{3}{10}$ths of an inch distant from the interior of the ring: this being carefully ascertained, the same was repeated in an exhausted receiver; the results are as follow:

TABLE II.

Number of Vibrations from 45° to 10°.			
In air.		In exhausted receiver.	
Vibrations in space.	Vibrations in ring of wood.	Vibrations in space.	Vibrations in ring of wood.
232	210	420	308

It may be seen in this experiment that the ring of wood exerted a very sensible retarding power on the vibrations of the bar, which is more apparent when the impediment to motion arising from a resisting medium is removed. I submitted to experiment in this way, in a pneumatic vacuum, a few rings of

* A magnetic bar whilst vibrating under the influence of any substance, must be considered to be operated on by two retarding forces; one by which it would be eventually brought to rest, supposing the given substance away; and another resulting from induction. The number of vibrations in a given arc, or their differences, therefore do not seem to be a measure of the latter force, since some portion of the former will always enter into the result of the experiment.

different substances similar to the above, and found their influence on the vibrations very decided, although the differences in their respective magnetic energies was not very apparent. The substances tried, with the respective number of vibrations, and their comparative energies as deduced from the formula, are given in the ensuing Table.

TABLE III.

Substances.	Vibrations from 45° to 10°.	Comparative energies as deduced from formula.
Distilled water, temp. 20° FAHR.....	330	0.27
Honduras mahogany	308	0.36
Statuary marble	306	0.37
Freestone	308	0.36
Annealed glass, $\frac{1}{10}$-inch thick	310	0.35

18. By employing the above substances in larger masses, the effect on the bar seemed to be greatly increased: although the difference in the comparative energies of wood and marble, the only two substances which I had an opportunity of trying, is still not very great. In the following Table are given the results of some experiments with a ring of statuary marble of 1.75 inch thick, and with a similar ring of Honduras mahogany, as compared with the influence of two rings of the same substances of only one fifth of an inch in thickness.

TABLE IV.

Substances.	Wood.		Marble.	
Thickness of ring in inches	0.2	1.75	0.2	1.75
Vibrations from 45° to 10°	308	280	306	265
Ratio of magnetic influence as deduced from formula	0.36	0.50	0.35	0.59

19. The influence of the substances being observed to depend in great

measure on the mass, it seemed of consequence to ascertain how far the energy might vary with the density. Several kinds of wood therefore of the above dimensions were subjected to experiment, whose specific gravities greatly varied; but there did not appear, after many trials, any perceptible difference on 280 vibrations. The woods tried were, Lignum Vitæ, Yellow Pine, African Oak, Honduras Mahogany, and Teak. All these reduced the vibrations of the bar from 420, the number performed in free space, to 280, the number performed when surrounded by the wood; the vibrations being taken from 45° to 10°, in an exhausted receiver.

20. I endeavoured to detect the magnetic energy of fluid bodies by vibrating the bar in an exhausted receiver, with its poles within the $\frac{1}{10}$th of an inch of the glass; the receiver being surrounded at that part opposite the pole of the magnet by the particular fluid, the subject of experiment. In this case the fluid was retained in its situation by means of an external check of glass ground to a glass-plate, on which the whole was sustained; the plate being supported on a ring of glass placed over and in connection with the pump-plate below; and the surfaces of contact ground, so as to fit air-tight. I failed however in arriving at any decided result; the amount of the retardation with sulphuric acid, for example, not appearing greater than was fairly attributable to the influence of the glass alone: even a saturated solution of sulphate of iron did not seem to exert in this way any sensible influence. By means of a very great number of vibrations, and a powerful bar, however, the energy of fluids, if such exist, might possibly become appreciable.

21. The inductive influence on metallic bodies being the most decisive, they are perhaps better adapted to the purposes of experiments relating to the influence of mass and density, or to the law according to which the inductive effect increases, as the distance from the magnet decreases, than any other kind of substance. Before examining, therefore, the comparative energies of various metals, I endeavoured to arrive at some further conclusions in this interesting part of the inquiry. In order to examine the comparative energies when the quantity of matter was varied, twelve concentric rings of copper were employed similar to the former ring of wood (17), except in thickness, which amounted to .025 of an inch: these rings were turned up with sufficient accuracy, and fitted very fair and close one within the other. The num-

ber of vibrations was first determined in the interior ring alone, after which a second was added, and the number of vibrations again determined, and so on to eight rings; when the increased effect became almost inappreciable.

TABLE V.

Number of rings	1	2	3	4	5	6	7	8
Vibrations	44	30	24	21	19+	17	16+	16
Ratio of force.	8.54	13	16.5	19	21+	23	24+	25

The increments in the effect on the bar appear by the above Table continually to decrease, so that after a certain number of laminæ they seem to be no longer appreciable; hence the numbers expressing the comparative energies would become at last equal, and such was found to be the case, the number of vibrations with twelve rings being no greater than with ten*.

22. The quantity of matter in the foregoing experiment being actually divided into concentric rings, I endeavoured to ascertain how far the above results depended on the sum of the whole, or of any number of them; with this view the number of vibrations was determined for each ring, when opposed to the bar in the same relative situation which it occupied in the mass, so as to obtain the separate effect due to any particular ring. These separated results were as follow.

* It is not improbable that some principle of this kind is the immediate cause of the curious result arrived at by Professor BARLOW in the course of his magnetic inquiries; who observed that a hollow sphere of iron exerted the same influence on a compass needle, as if it were a solid mass. If we consider the sphere to be made up of concentric laminæ, then, by the foregoing experiment, the number of laminæ equivalent to exhaust the inductive energy of a small compass needle, will, especially in iron, (11) (Exp. *f* note,) be very few: consequently, all the interior may be removed. This is likewise consistent with a subsequent observation of the same distinguished philosopher; a certain depth of metal being found necessary. If the experiment were accurately tried, it is probable that the depth of metal requisite, would vary with the force of the compass bar; although for small needles the differences are so inconsiderable as not to be appreciable. For very powerful magnets, however, some further quantity of metal would be required; but, perhaps, in no instance could we employ a magnet of sufficient power to influence the whole mass of a solid iron sphere of large dimensions.

TABLE VI.

No. of the ring	1st	2nd	3rd	4th	5th	6th	7th	8th
Vibrations....	44	76	94	124	148	166	186	210
Ratio of force.	8.54	4.52	3.45	2.38	1.83	1.53	1.25	1.0

In comparing the above numbers with those in the preceding Table, the energy of any number of rings, taken together, appears to be very nearly the sum of the same rings, taken separately, and is a curious deduction by the formula above given (16). Thus the energy of three rings combined is by the preceding Table 16.5, which is about the aggregate amount of the first, second, and third, as deduced above, and so on. It is true that some of the other numbers are not in such complete accordance as in this instance, but the approximation is nevertheless very remarkable. The intervening rings do not therefore intercept any very sensible portion of the inductive influence of the bar; a result quite consistent with what is observed in the case of rotating discs, when covered with non-ferruginous screens; and with this further condition, that whilst the inductive energy thus penetrates the intervening metal, it disturbs at the same time its magnetic distribution.

23. The foregoing deductions were verified by combining a few rings not immediately successive, so as to leave an interval between them. The results are as follow.

TABLE VII.

Rings combined.	1st+4th	4th+8th	1st+4th+8th
Vibrations.....	−37	100	+33
Force.....	11.3	3.2	−11.7

In these, as in the foregoing instances, it may be perceived, on comparing the actual observation, as given in this Table, with the values of the respective rings in the preceding one, that the numerical approximations are very close. The energy of any ring therefore may be estimated by subtracting from the aggregate effect, the sum of the others with which it is combined.

24. It would not be difficult, from the above investigation, to arrive at some

general conclusions as to the precise ratio in which the energy increases with the separate quantities of matter, and decreases with the distance; since both are involved in the experimental results. In the following Table is given the observed effect of each separate ring, as compared with the corresponding effect deduced by calculation from the first experiment, on the supposition that the energy is directly as the quantity of matter, and inversely as the squares of the respective distances; and it will be seen that the differences are not greater than may, from the somewhat complicated nature of the experiment, be expected.

TABLE VIII.

Rings	No. 1.	No. 2.	No. 3.	No. 4.	No. 5.	No. 6.	No. 7.	No. 8.
Energy by calculation		4.5	3.05	2.19	1.65	1.28	1.04	0.84
Observed energy	8.54	4.52	3.45	2.38	1.83	1.25	1.25	1.0

25. There appears reason therefore to conclude, as a general principle, that the transient magnetic energy of any substance, not supposed to contain iron, varies in a direct ratio of the quantity of matter within the sphere of the action, and in an inverse ratio of the squares of the distances from the magnet; the matter being supposed to be condensed into a stratum not having sensible thickness, and taken at some mean point of distance within the surface, where the sum of the forces may be imagined to produce the same effect as if exerted from every part of the mass; and the respective distances estimated between this mean distance and the opposed surface of the magnet*.

26. It is a consequence almost necessarily resulting from the foregoing investigation, that, supposing all other circumstances alike, the energy of the same substance is directly as its density, although such did not actually appear in the foregoing experiments with various species of wood (25); but this may be readily supposed to arise from the circumstance, that the energy of wood is

* The reason why such should be the case for the magnet has been already given (12): it is equally applicable when the energy of the extremities only is supposed to be exerted: and it may be further observed, that although on either of the sides of a magnetic bar, the point of greatest attraction of that particular side seems to fall (a little) within the extremity, yet the maximum of force of every common bar will be found invariably on the surface, terminating the extreme ends.

altogether too feeble to admit of such differences becoming appreciable by the means employed. In metallic bodies, however, the fact appears to be very decided; thus, in three specimens of copper, whose dimensions were very nearly the same, but whose specific gravities varied, the resulting energies were observed to be the greatest in those which had the greatest specific gravity; as for example, when the specific gravities were to each other as 8.30 to 8.40, the resulting number of vibrations amounted in the one instance to twelve, in the other to fifteen. To obtain more than rough approximations to the precise law by this method, would be extremely difficult, since the metals, as it is evident, should be procured in a state of great purity, and the dimensions, previously to taking the specific gravities, adjusted with an accuracy which only the most refined workmanship can effect; there is however little doubt, if the experiment could be accurately tried, that the energy of the same metallic substance would, as a general result, be in the direct ratio of its density.

27. It is of importance therefore, in estimating the comparative magnetic influence of metals, to take into account their density; without which the inquiry might prove unsatisfactory. The energies of various metallic bodies, and some few alloys, were estimated by determining the number of vibrations of the bar between 45° and 10° in a ring of each; the metal being within 0.1 of an inch of its poles. The more common metals were cast in a mould, and all the rings adjusted to the same dimensions.

28. I endeavoured to estimate the energy of mercury at −50° of Fahrenheit, by freezing it into a solid ring, of the same dimensions as the preceding, in a mould of wood, accurately adjusted for the purpose; but could only obtain satisfactorily, as a general result, the place it appeared to hold in the scale of magnetic energy. The ring was fully and completely solidified, and the mould so constructed as to admit of its interior part being removed, so that the needle could oscillate, whilst the metal remained externally enveloped in the freezing mixture. The thermometer, previously to removing the interior of the mould and adjacent salts, stood at −56°. I had not an opportunity of repeating this interesting experiment, which is sufficiently practicable, and seems to promise the development of some new phenomena in magnetism. The energy of mercury in its fluid state, in a vacuum, was examined by inclosing it be-

tween two glass circular checks, so as to obtain a ring of nearly the same dimensions as the others, subsequently separating the influence due to the glass alone.

29. The place which the following metals appear to occupy in the scale of magnetic energy, and their comparative influence, as resulting from this investigation, is given in the succeeding Tables.

TABLE IX.

Metals.	Rolled Silver.	Rolled Copper.	Cast Copper.	Rolled Gold.	Cast Zinc.	Cast Tin.	Cast Lead.	Solid* Mercury.	Cast Antimony.	Fluid Mercury.	Cast Bismuth.
Comparative magnetic energy	39	29	20	16	10	6.9	3.7	2.0	1.3	1.0	0.45

TABLE X.

Alloys.	Cast Copper and Zinc in equal parts.	Cast Copper and Bismuth in equal parts.	Cast Zinc and Bismuth in equal parts.
Comparative energy ..	12	2.3	1.4

30. Although considerable pains have been taken to make the foregoing Table as perfect as possible, yet it cannot be considered as anything more than a useful approximation; there are many conditions peculiar to this inquiry to be yet investigated, which seem for the present to preclude the possibility of obtaining results quite conclusive. The metals employed were in as great a state of purity as they could be obtained in the ordinary way of commerce; some of them, more especially the copper, gold, and silver, may be considered as very nearly without alloy.

31. If the preceding investigations are of any importance, they seem to be in great measure conclusive, as to the cause of the influence of a rapidly

* The comparative influence of solid mercury must be taken as a rough approximation; the bar in this ring in air, as nearly as could be ascertained, performed about 150 vibrations between 45° and 10°; whilst, in an analogous ring of rolled copper it performed about fourteen vibrations; in free space in air, it performed 232.

rotating metallic disc on a freely suspended magnet; whilst at the same time they are in accordance with the opinion of M. Arago, that almost every known substance can exert an influence on the compass-needle. Every species of matter therefore may be considered to be more or less susceptible of a state of transient magnetic energy, arising from induction.

32. The hypothesis advanced by Mr. Herschel and Mr. Babbage in explanation of some of these phenomena is extremely simple, and to a great extent very satisfactory; inasmuch as it agrees with all the observed operations of magnetic induction, and, as stated by them, supersedes the necessity of advancing any new hypothesis in magnetism; it is nevertheless considered by M. Arago as insufficient*. This celebrated philosopher appears to deny, that the limit of the motion produced by any force, which can be supposed to reside in the induced poles, even if their existence be admitted, can exceed a minute of a degree: whereas, in order to explain the rotation, it should exceed 90°. It may be observed however, that a ring of metal, copper for example, immediately surrounding the poles of the needle, can, as first stated by M. Arago, diminish the amplitude of its oscillations, and, as seen in the foregoing experiments, so fetter its motion, as to reduce the vibrations in a given space from 420 to 14. We must therefore necessarily conclude the force induced in each consecutive point of the ring to be very considerable. With non-metallic bodies, the force is certainly very much less; but these do not fetter the motion of the needle to anything like so great an extent, nor will these substances rotate, as far as I can find, by the influence of a magnet revolving in an exhausted receiver without sensible vibration, notwithstanding the rapidity of the rotation exceeds 700 revolutions in a minute.

33. After considering with much attention the hypothesis just alluded to, I am led to offer a few additional observations, with a view of extending the principle it involves, and which may possibly be useful in further elucidation of the perceived effects; whilst, at the same time, they will not be found in any way inconsistent with the known laws of magnetic action.

The facility with which a magnetic development is induced in bodies, may be designated by the term "susceptibility"; the time during which this effect of induction remains in them "retentive power". These terms have been

* Edin. Philos. Journal, vol. v. p. 326.

already employed by Mr. Babbage and Mr. Herschel, and they are sufficiently explanatory. In the case of magnetic energy induced in a body by any of the ordinary methods, it may be observed as a general fact, that the energy is acquired in somewhat less time than that in which it is again lost. This is particularly the case in soft iron rendered magnetic, by an electro-magnetic spiral; a simple contact between a magnet and a mass of soft iron will frequently convey to it an attractive force, which it retains for a comparatively long space of time. In regard to the susceptibility of different substances, it is found to vary considerably; and depends on some peculiar property not yet explained; it seems to be in some inverse ratio of their retentive power. Thus it is not without difficulty that hardened steel is made permanently magnetic, whilst its retentive power is considerable: soft iron, on the contrary, is observed to be comparatively feeble in its retentive power, whilst its susceptibility of magnetic change is great. It may be hence inferred, that in non-ferruginous masses, the mere susceptibility of magnetic change, is in fact also considerable, but then their retentive power is so feeble, that the subsequent attraction does not ensue to any great extent. This probably arises from some peculiar state of the particles of these substances, which allows what may be termed the new magnetic distribution, to tend more rapidly to the previous state of neutrality, immediately the tension passes a certain point. In every observed instance, however, more time seems necessary to restore perfectly, the original state of the body, than was required to disturb it, the former being the result of a progressive operation, whilst the latter is effected by a sudden and concentrated force.

34. The attraction, as usually observed, between a magnet and a mass of iron, is invariably preceded by this new magnetic distribution in the iron; and unless such new distribution can occur, it seems, as in the analogous operation of electrical action, that no attractive effect can ensue; indeed this is made evident by the repulsive efforts of two similar magnetic poles; the repulsion may be conceived to be really the immediate consequence of an inductive effort, since an attempt is made to reverse the magnetic distribution in the opposed bars, which action the already existing polarities resist: when, however, as in the case of opposing either pole of a very powerful

magnet to the similar pole of a very weak one, this new distribution can be effected, an attractive force immediately follows.

35. There are some circumstances connected with this curious result of magnetic action, which seem to apply immediately to the question under consideration, but which have not been generally observed. If two similar magnetic poles of very unequal force be opposed to each other, the greatest repulsion, taking into account the difference in distance, will be a little within the limit of their action: that is to say, the increments in the repulsive energy are comparatively less, as the magnets approach each other: evidently resulting from the change which begins to take place in the magnetism of the bars. Now if one of the bars be extremely powerful in regard to the other, the new distribution in the weaker bar is effected even before the point of contact. The precise point at which the existing polarity of the weaker bar becomes changed varies with the force of the stronger magnet. This point may be found by experiment in the following way. Let a small cylindrical magnet be suspended by a silk line from a delicate wheel, whose axis rests on friction rollers; and let it be counterpoised by a small weight at the extremity of a short cylindrical piece of wood partly immersed in a jar of water. If one pole of a powerful bar be now carefully approximated towards the like pole of the suspended magnet, by fixing the former in a brass frame carrying a micrometer-screw, the latter will be observed to recede from the bar, until the opposed poles are within a certain distance of each other; when the repulsion will cease, and a weak attractive effect ensue. The cylinder of wood as it becomes gradually immersed *, continually furnishes an equivalent to the repulsion, in the quantity of water displaced.

We do not generally perceive the attractive effect resulting from this inductive action in non-ferruginous masses; for, as already observed, their feeble retentive power admits of the magnetic neutrality being more rapidly restored, when the tension passes a certain point. So that in fact, the opposite magnetic state never becomes sufficiently intense to evince an attractive force, cognizable by the ordinary means.

* A more detailed account of this experiment may be found in the Transactions of the Royal Society of Edinburgh.

36. When, therefore, a disc of any substance is put into a state of rotation under a suspended magnetic bar, opposite poles are induced in that part of the disc immediately under the bar; these induced poles may be supposed to pass on, and being retained for a short portion of time, will, if the motion be more rapid than the time during which the impression remains, be transferred toward the opposite poles of the magnet, and exert upon them a repulsive action up to that point of distance (35) at which the poles of the magnet again reverse the transient poles impressed on the disc, and substitute opposite poles, to be again reversed as before.

37. Thus if *N s S n* (fig. 12) be a metallic disc revolving in the direction *N s S n* under a magnetic bar *N S* free to move upon a central point *C*, and of which *N* is the north pole, and *S* the south, the effect of this rotation will be, to impress upon the semicircle *N s S* a south polarity, and upon the semicircle *S n N* a north polarity, in consequence of the points *s s s* and *n n n* having passed under the poles *N* and *S*. Now if the time required to restore the original magnetic distribution of the plate, be less than that necessary to disturb it, these impressed polarities remain for a small portion of time, and hence there will always be an attractive force in advance of the poles of the bar, and a repulsive force in arrear of them; consequently the bar becomes driven, as well as dragged in the direction of the revolving disc by the resolved portion of the oblique actions acting for the most part near the extremities. And there is little doubt, that any substance in the least degree susceptible of a transient magnetic state, might cause a magnet to rotate; provided that the rapidity of the motion be greater than the time necessary for the restoration of the original magnetic distribution of the body acted on; supposing such rapidity of rotation possible.

38. A rotating disc, therefore, circumstanced as above (37), may be considered as a circular magnet such as that already mentioned (5), one of the semicircles having a north polarity, and the opposite semicircle a south polarity; and which polarities eventually neutralise each other about one of the diameters; the only difference being, that the magnetism of the revolving disc is transient, and constantly changing its position; so that the neutral points are always near the poles of the suspended bar. The bar therefore by a well-

known law of magnetic attraction, will be constantly endeavouring to place itself at right angles to the magnetic equator of the disc, whilst the position of the latter varies in its turn with the position of the bar: it is in fact owing to this circumstance that the bar at length revolves with the disc, as may be in great measure seen by substituting for the disc a similar disc of steel made permanently magnetic; in which case the suspended bar will not revolve, it will be merely put into a state of disturbed vibration.

V. *On the Nature of Negative and of Imaginary Quantities.*

By DAVIES GILBERT, *Esq. P.R.S.*

Read November 18, 1830.

I AM desirous of submitting to the Royal Society some observations on the nature of what are termed Negative and Imaginary Quantities, tending as I hope to clear away the obscurity that has hitherto surrounded them.

The subject has occupied my attention for many years, and however plain and simple may be the results, they have not been obtained without much patient investigation; and, in the event of their being found correct, they will add one authority more to an observation frequently made, and confirmed by extensive experience,—That paradoxes and apparent solecisms, involving themselves with facts or with deductions known to be true, may always be found near the surface, owing their existence either to ambiguities of expression or to the unperceived adoption of some extraneous additions or limitations into the compound terms used for definition, which are subsequently taken as constituent parts of their essence.

The first misapprehension appears to consist in our considering any quantity whatever as negative per se, and without reference to another opposed to it, which has previously been established as positive.

In applying our arithmetic to whatever is continuous, some neutral point or zero must be selected, as in time, in space with reference to its three dimensions, in forces, in velocities; and the opposite directions from this point will be mutually negative in respect to each other, and must be distinguished by appropriate marks or signs. But space to the left is no more essentially negative than space to the right, nor descent than ascent, nor time past than time that is to come.

I would therefore adopt for the present investigation, and to avoid previously formed association of ideas, (*a*) and (*b*) as marks or signs for prefixing

to the same quantity taken in opposite directions, rather than the usual ones of plus and minus.

In the next place, I believe that the law of the signs has never been stated according to its full generalization.

In common language, and for ordinary purposes, multiplication is considered as an abbreviated addition; but it would be a superfluous waste of time to demonstrate before this Society that multiplication is always an affair of ratios. Length and breadth multiplied together give areas, because an unit of length by an unit of breadth has previously been established as the superficial unit. Length, width, and depth give solidity for the same reason, and from the want of such a preestablished unit, arises the utter absurdity of a question, heretofore proposed in various treatises on arithmetic, for multiplying some denomination of coin by itself, and ascertaining the product.

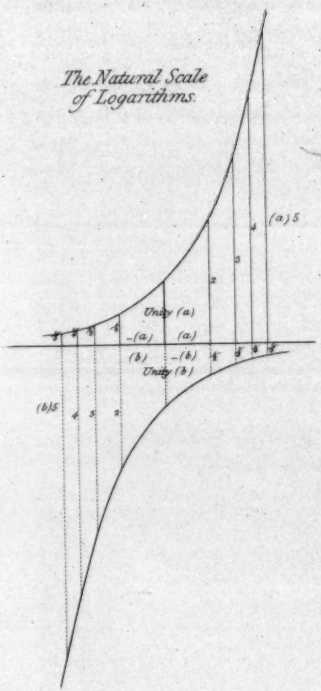
When a multiplication of two quantities is therefore to be made, unity must be understood as the antecedent; but here an extraneous limitation insinuates itself unperceived. The common antecedent taken in usual practice is not simply an unit, but unity in the scale of (a). With this limitation the law of the signs is correct, namely that like signs produce (a), and unlike signs produce (b). But let unity, the common antecedent, be taken in the scale of (b); the law will then immediately change to like signs producing (b), and unlike signs producing (a).

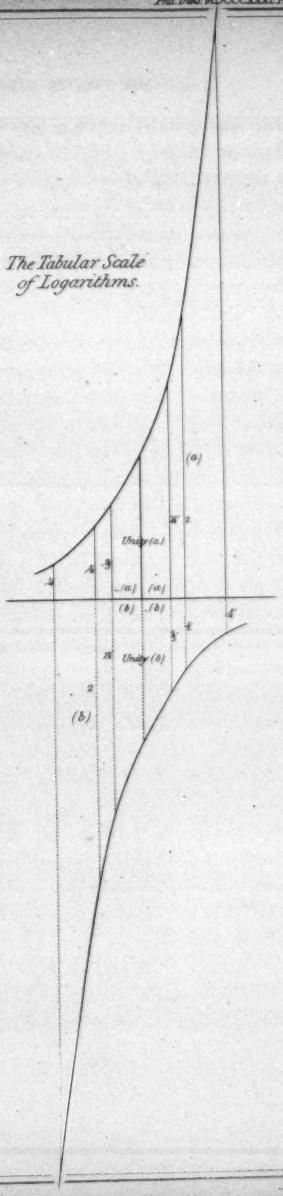
The general rule therefore must be, that like signs give the sign of the assumed universal antecedent, and unlike signs the contrary.

Admitting, therefore, that both scales are in themselves equally affirmative, and that either may be taken as negative to the other, it necessarily follows that by using the scale of (b), and consequently by assuming the unit of that scale as the universal antecedent, all even roots in the scale of (a) will become imaginary; and thus the apparent discrimination of the two scales is entirely removed: and in the same way, and by varying the signs according to the scale in which the universal antecedent is taken, all formulæ will become equally applicable to both.

For example: (See plate III.)

The natural numbers and their logarithms will be expressed for both scales by the correlative curves in the following figures, where all ordinates taken


The Natural Scale of Logarithms.
(a) 5
4
3
2
Unity (a)
-(a)
(a)
(b)
-(b)
Unity (b)
(b) 5
4
3
2


The Tabular Scale of Logarithms.
(a)
2
Unity (a)
-(a)
(a)
(b)
-(b)
Unity (b)
2
(b)

continuously above and below the line of the apsides are reciprocal to each other; and their product equal to the square of either ordinate at the zero point, which, in the position of the curves corresponding to the natural system, is when the fluxions of x and y are equal.

It is obvious from inspecting this figure, that if m be a natural number and M its logarithm, both in the scale of (a), and M be taken in the scale of (b), or $-(a)$, it will become the logarithm of $\frac{1}{m}$; but M in the scale of (b) is also the logarithm of m in the same scale; and M taken as at first in the scale of (a), or $-(b)$, is the logarithm of $\frac{1}{m}$.

If the two curves are moved on each other, so that the two ordinates measuring 2.302585 in reference to the former unit be made continuous, and that length be taken as a new unit, they will then represent the tabular logarithms in both scales.

I flatter myself with having now clearly established the principle, that all properties belonging to the scales of (a) and (b) are mutually interchangeable; and that consequently imaginary quantities will be found in the even roots of either scale, as the universal antecedent is taken in the other. And this leads to the question,

What are imaginary quantities?

I answer, Creations merely of arbitrary definition, endowed with properties at the pleasure of him that defines them, the whole dispute respecting imaginary quantities turning on the point contested from the earliest times between the hostile sects of Realists and Nominalists, descending through Plato, Aristotle and the Stoics, to the Philosophers of Alexandria, and from them to the Schoolmen, who imbittered their discussions with theological controversy and persecution.

If the conceptions of the mind, all abstractions and generalizations, were considered as substantial forms, possessing existences distinct from the intelligences contemplating them; or, as some writers have expressed themselves autousia gaudentes, the μορφαι immutable and eternal, which united to matter constitute the universe, nothing could be equally absurd with the supposition of impossible or imaginary quantities; but according to the theory now so uni-

versally prevalent as scarcely to admit of a single exception among all those who make the powers of the human mind the subject of their peculiar research; Classifications, Abstractions, Generalizations are allowed to be mere creatures of the reasoning faculty, existing nowhere but in the mind by which they are contemplated. To such unsubstantial existences any qualities may be imputed; but the only one known or useful in algebra, is the supposed even root of a real quantity taken in the scale opposite to that which has given the universal antecedent. The sign or mark indicating the extraction impossible to be performed, veils the real quantity, and renders it of no actual value until the sign is taken away by an involution, or the reverse of the supposed operation which that mark or sign represents, although by its arbitrary essence the quantity so veiled is in the mean time made applicable to all the purposes for which real quantities are used in all kinds of formulæ.

While therefore the sign of the supposed extraction of a root remains, the quantity to which it is prefixed has no more than a potential existence; but it stands ready to exist in energy whenever that sign is removed.

Consequently, without experience, it is impossible to know whether an implicit function of such an ideal quantity, will or will not be cleared by development of the symbol indicating the supposed extraction of a root, that is, whether any actual value does or does not belong to such a function.

Subject to the above conditions, namely, that the quantity veiled by the sign of a supposed extraction shall be treated in expansions and formulæ according to the laws applicable to real quantities, and that it shall exist in energy whenever an involution has reversed the supposed extraction of an even root,—

Let (A) be supposed equal to $\sqrt{-1}^{\sqrt{-1}}$ to find (A);

Then, according to the established laws of real quantities arbitrarily extended to these that are imaginary, the log. of $A = \sqrt{-1} \times$ log. of $\sqrt{-1}$; but by a well known theorem,

$$\text{the log. of } \sqrt{-1} = \left(\sqrt{-1} - \frac{1}{\sqrt{-1}}\right) - \tfrac{1}{2}\left(\sqrt{-1}^{2} - \frac{1}{\sqrt{-1}^{2}}\right)$$

$$+ \tfrac{1}{3}\left(\sqrt{-1}^{3} - \frac{1}{\sqrt{-1}^{3}}\right) - \tfrac{1}{4}\left(\sqrt{-1}^{4} - \frac{1}{\sqrt{-1}^{4}}\right) \&c.$$

And this series $\times \sqrt{-1}$ becomes $-2 \times \frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9}$ &c. each alternate term

vanishing. But $2 - \frac{2}{3} + \frac{2}{5} + \frac{2}{7} - \frac{2}{9}$ &c. = the quadrantial arc of a circle to radius unity.

Therefore the log. of A = − quad. arc.

$$\text{And } A = e^{-\text{quad. arc}} = 0.2078796.$$

Consequently, $\sqrt{-1}^{\sqrt{-1}}$ is an abbreviated mark or symbol, according to the above arbitrary conditions, for the radix of the natural system of logarithms raised to a negative power, indicated by the quadrantial arc of a circle to radius unity. And in the event of $\sqrt{-1}^{\sqrt{-1}}$ ever occurring in the solution of a problem, $e^{-\text{quad. arc}}$ or $2.71828^{-1.5708}$ or 0.2078796 may be substituted for it. And this is what practically happens in regard to all expressions apparently imaginary, which are found to represent real quantities, as is well known in cases of circular arcs and logarithms. These mental abstractions have moreover extended the bounds of analysis far beyond the utmost limits it could otherwise have attained; they have bestowed harmony and simplicity of form on its most recondite investigations, and eminently has their use been important in equations, by resolving them into a number of simple factors equal to the dimensions of the equation in its highest term.

It appears from these considerations, that several ingenious mathematicians have taken an incorrect view of ideal quantities, by mistaking incidental properties for those which constitute their essence; as, for example, when they are supposed to be principles of perpendicularity, because they may in some cases indicate extension at right angles to the direction here designated by (*a*) and (*b*), but with an equal degree of propriety might the actually existing square root of a quantity be considered as the principle of obliquity; because, in certain cases, it indicates the hypothenuse of a right-angled triangle.

I would here notice an error in reasoning (as it appears to me), fallen into by all authors who have endeavoured to explain the mode of arriving at a true conclusion respecting the sines and cosines of multiple arcs; which reasoning imputes actual properties to ideal quantities, instead of deriving them all from mere arbitrary convention.

Given the sine, and consequently the cosine of an arc, to find the sine and cosine of n times that arc:

Let z the original arc, v the sine, and y the cosine, x the cosine of the arc $n\,z$, then as $\dot{z} : -\dot{y} :: 1 : \sqrt{1-y^2} \therefore \dot{z} = \frac{-\dot{y}}{\sqrt{1-y^2}}$ therefore

$$\frac{\dot{x}}{\sqrt{1-x^2}} = n \cdot \frac{\dot{y}}{\sqrt{1-y^2}}$$

No integration can however be effected of these quantities in their actual form; but changing the signs of the terms in both denominators,

$$\frac{\dot{x}}{\sqrt{x^2-1}} = n \cdot \frac{\dot{y}}{\sqrt{y^2-1}} \text{ and}$$

the nat. log. of $(x + \sqrt{x^2-1}) = n \times$ the nat. log. of $(y + \sqrt{y^2-1})$

and $x + \sqrt{x^2-1} = (y + \sqrt{y^2-1})^n = y^n + n\,y^{n-1} \cdot \sqrt{y^2-1}$

$$+ n\frac{n-1}{2} y^{n-2} \cdot y^2 - 1 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot y^{n-3} \cdot \dot{+}\sqrt{y^2-1} \text{ \&c.}$$

But since y is taken as the cosine of the original arc, and x is the cosine of the multiple arc, and consequently each less than unity, it is obvious that the second term on the left of the equation, and that every even term of the expansion on the right, can exist only in the potential form of an ideal quantity; and a conclusion has thence been drawn (but as it seems to me on no solid principle whatever), that since real and imaginary quantities occur on each side of the equation, and they are of a nature completely heterogeneous one to the other, each must be respectively equal; but this mode of reasoning clearly imputes qualities to mere symbols beyond those originally imparted to them. The double equality, on my principles, depends entirely on its being assumed; as in the solution of cubic equations.

When $a + b$ have been substituted for x in the equation $x^3 - q\,x + r = 0$, and it becomes changed into $a^3 + b^3 + (3\,a\,b - q) \times \overline{a+b} + r = 0$, two unknown quantities exist with but one relation; another may therefore be assumed; and that which obviously reduces the expression to the most simple form is obtained by making $3\,a\,b - q = 0$.

In the same manner x and y, the cosines of two arcs, having but one relation, admit of another being assumed; any relation might be taken, but the one clearly indicated is that which makes the real terms on both sides equal:

this assumption leaves the ideal quantities in their actual state without any change, and when the sign of an imaginary operation has been removed in the usual way, they also become truly equal on each side of the equation, and from this double equality, series readily present themselves expressing the sine and cosine of the multiple arc required.

On the whole, it appears to me that mystery and paradox are entirely banished from the science most powerful in eliciting truth, and where they ought least to be found,—by considering all quantity as affirmative per se, and admitting plus and minus only as correlative terms; consequently, by extending the law of the signs, so as to make the multiplication of like signs give that of the scale in which the universal antecedent has been assumed, and the multiplication of unlike signs the contrary; and, finally, by excluding all actual existence, and thereby all inherent properties, from the symbols of quantities veiled by the mark indicative of an operation incapable of being performed, but arbitrarily endowing them with the properties of real quantities in all expansions and formulæ, and with the ultimate quality of regaining their actual existence whenever the veil is removed, by an operation the inverse of that which had originally induced it.

VI. *On the probable Electric Origin of all the Phenomena of Terrestrial Magnetism; with an illustrative Experiment.* By PETER BARLOW, *Esq. F.R.S. Cor. Mem. Inst. of France, of the Imperial Academy of St. Petersburgh, &c. &c.*

Read January 27, 1831.

IN order to show the incompatibility of the observed laws of terrestrial magnetism with the supposition of the earth itself being a magnet, and at the same time their accordance with the laws which appertain to a body whose magnetism is induced by electricity, it will be necessary to take a retrospective view of the several discoveries which have been made, connected with these subjects, since the commencement of the present century, and particularly within the last ten or twelve years. Up to the period of the scientific travels of M. HUMBOLDT, it must be admitted that all the facts with which we were acquainted, relative to terrestrial magnetism, were a mere collection of observations and phenomena, uncertain in themselves, unconnected with each other, and irreducible to any specific laws; but the confidence inspired by the high character of this distinguished traveller, his accuracy of observation, and the perfection of his instruments, gave a new feature to the inquiry, and laid the foundation of our present knowledge in this science.

M. BIOT was the first to undertake the difficult task of reducing these results to some principle of calculation. Considering the earth as a magnet, he assumed an indeterminate distance to represent the distance of its two poles; and then, supposing their power to vary inversely as the square of their distance from the point on which they acted, (a law which had been already established,) he obtained a general expression for the direction of a magnetic needle. He then made his indeterminate distance vary; and comparing at every step his results with those observed, it was found that the nearer the poles were made to approach, the nearer the computed and observed results corresponded; and finally, that the errors were reduced to a minimum when the two poles were coincident, or indefinitely near to each other.

This is a highly important determination as it relates to the present inquiry; for here we have at once a demonstration that the earth is not a magnet, or at least that it does not act according to the laws of a permanent magnetic body, the distinguishing characteristic of which is, the existence of two poles, distinct and distant from each other; whereas it is shown by this investigation, that in the earth these two poles are indefinitely near to each other and to the earth's centre.

M. Biot's results, however, in consequence of the generality he had given to his first assumption, were involved in a very intricate formula; but in the mean time a similar task had been undertaken by M. Kraft, of St. Petersburgh, on different principles. This philosopher contented himself with attempting to determine empyrically some mathematical formula to connect the different dip observations with each other; and he discovered this very simple relation, viz. "That the tangent of the dip of the needle in any place is equal to double the tangent of the magnetic latitude of that place." M. Biot, in consequence of this deduction, re-examined his former investigation, and found that his formula after certain reductions, of which he was not before aware it was susceptible, became identical with that of M. Kraft; and thus the fundamental law of terrestrial magnetism was confirmed by two philosophers acting apart and independently of each other, and on principles entirely different. It followed also from M. Biot's formulæ, that the intensity of the dipping needle ought to vary inversely as the square root of 4 *minus* 3 times the square of the sine of the dip; and that of the horizontal needle inversely as the square root of 3 *plus* the square of the secant of the dip: conditions which have since been verified, at least approximatively, by observations, extending even to within one or two degrees of the pole itself. And these laws, I beg again to repeat, are entirely inconsistent with those which appertain to a permanent magnetic body, while they will be shown, in what follows, to be the fundamental laws of a body which receives its transient magnetism by induction. We see thus how it happened, that those philosophers of the last century, who endeavoured to illustrate the laws of terrestrial magnetism by what they called terrellæ, or round natural magnets, with distinct poles, failed in all their attempts on this subject to establish any mathematical principle of calculation.

It has been seen that up to the year 1809, the phenomena of terrestrial mag-

netism had resisted every prior attempt to reduce them to the power of analysis. It will now be necessary to refer to those which relate to the reciprocal action of iron bodies and a magnetic needle on each other, which were still more uncertain, and apparently more anomalous, than the former.

In 1819 I undertook this investigation, and, after an extensive series of experiments, I succeeded in reducing this reciprocal action likewise to very simple laws, derived empyrically from the results I had obtained, but which were obviously only approximative, although the approximation was very close: in the course of these experiments also a very remarkable fact was discovered; namely, that all the magnetic power of an iron sphere resides on its surface. These first experiments, with the exception of the fact last mentioned, which I had not then discovered, were presented to the Royal Society in 1819. I afterwards extended my first researches, and published the whole in a small tract, entitled, "An Essay on Magnetic Attraction."

The simplicity of these results, and particularly the circumstance of the magnetic power residing on the surface only of the sphere, led Mr. CHARLES BONNYCASTLE, (at present mathematical professor in the University of Virginia,) to undertake an investigation of the laws which an iron sphere ought to exhibit according to a certain hypothesis relative to induced magnetism; and he succeeded in eliciting most of the formulæ I had obtained empyrically from my experiments. In the second edition of my Essay, I also examined the subject analytically, making a slight alteration in the hypothesis employed by Professor BONNYCASTLE; and by this means have been enabled to deduce all my experimental laws without exception, and to supply the small corrective part which was obviously wanted in the several formulæ as they were first obtained. Since that time, M. POISSON has employed his powerful analysis to investigate the subject in all its generality; and I have had the satisfaction of seeing confirmed by so distinguished a mathematician all my original propositions; and an action considered till that time anomalous, reduced to laws nearly as general and certain as those which govern the planetary motions.

The application of these deductions to the present inquiry is important. It follows from these laws, that if an iron sphere, such as we have supposed in a transient state of magnetic induction, be made to act upon a magnetic needle, isolated from the terrestrial magnetism, it will produce in that needle all the

phenomena which M. Humboldt observed, and which MM. Biot and Kraft reduced to determinate laws of action *; the two poles in this case, as in that, being indefinitely near to each other, and to the centre of the sphere. Hence then it follows,

1. That the laws of terrestrial magnetism are inconsistent with those which belong to a permanent magnetic body.

2. That they are perfectly coincident with those which appertain to a body in a transient state of magnetic induction.

These results were incontrovertible; but an insuperable obstacle seemed still to oppose itself to any rational hypothesis relative to the cause of the earth's magnetic power. Up to this period we knew of only one means of inducing magnetism, which was by the approximation of a permanent magnet to a ball or mass of simple iron, and one or two other metals: but what body could be imagined capable of inducing this power in the earth; particularly as the earth preserved its magnetic energy constantly in nearly the same direction, whereas its position with regard to any exterior body was hourly changing? Its magnetism could not therefore be induced by any foreign body; and as no

* The formula indicating the position of a magnetic needle freely suspended from the combined action of the earth and an iron sphere upon it, is

$$\tan \Delta = \frac{3 \cos \varphi \cdot \sin \varphi}{\frac{d^3 M}{r^3 C} + 3 \cos^2 \varphi - 1}$$

where Δ is the deflection from the axis of the sphere, φ the complement of the magnetic latitude, M the magnetic power of the earth, r the radius of the iron sphere, d the distance of the needle from its centre, and C a constant coefficient dependent on the magnetic power of the iron. In this expression making M vanish according to the above supposition, and substituting $\varphi + \delta'$ for Δ, so that δ' becomes the complement of the dip, we have

$$\tan (\varphi + \delta') = \frac{3 \cos \varphi \sin \varphi}{3 \cos^2 \varphi - 1}$$

which after an easy reduction becomes

$$2 \tan \delta' = \tan \varphi, \quad \text{or} \quad \tan \delta = 2 \tan \lambda$$

where δ is the dip, and λ the magnetic latitude. By a similar process, calling I and I' the intensity of the dipping and horizontal needle, we find

$$I = 2 A \sqrt{\frac{1}{4 - 3 \sin^2 \delta}} \quad \text{and} \quad I' = 2 A \sqrt{\frac{1}{3 + \sec^2 \delta}}$$

which are precisely M. Biot's formulæ.—See Essay on Magnetic Attractions, page 195.

power in itself could be conceived capable of producing this effect, the deductions above made, although they would have stood incontrovertible, yet the causes of them would have remained inexplicable, but for the important discovery of Mr. Oersted, which threw a new light upon magnetic investigations.

This philosopher discovered that a wire conducting an electric current was, during the interval of transmission, in a state of magnetic induction. Such a discovery, at such a time, was most fortunate; and not on this point only, but on various others, it excited the attention of all the most distinguished philosophers of Europe. The number of interesting facts which were thus elicited will always form a prominent feature in the scientific history of the nineteenth century, but the greater part of them are unconnected with the present inquiry; I shall therefore only refer to one or two, which have an important bearing upon the question under investigation. As soon as I was informed of this interesting discovery, I was anxious, by experiments as nearly similar, as circumstances would admit, to those I had adopted with the iron ball, to elicit in this case also the laws which govern the reciprocal action of the wire and the needle; and after a pretty long series of experiments I arrived at this conclusion, viz.—

"That the force of each particle in the wire on each particle of the needle varies inversely as the square of the distance, and that the nature of the force is tangential, that is, such as would place a needle, neutralized from the earth's magnetism, always at right angles to the direction of the wire, and to the direction of the line joining the needle with the centre of action of the wire." This law, with an account of the experiments from which it was derived, was read to the Royal Society, May 22nd, 1822, and was afterwards published in the second edition of my Essay above referred to.

While I was thus engaged in endeavouring to elicit the law of action between the wire and the needle, M. Ampere had entered upon a much more extensive investigation; that is, not only of the reciprocal action between the wire and the needle, but also between different wires and galvanic currents on each other. Galvanic needles, both dipping and horizontal, were constructed, which possessed all the properties of the usual magnetic needles. The law of action of galvanic currents on each other was reduced to that of attraction; and by

assuming the magnetism of the needle to be due to an infinite number of galvanic currents parallel to each other and at right angles to the axis, the action of needles on each other, of these on galvanic currents, these currents on each other, and of the earth itself on each, were all reduced to one general principle, admitting of accurate and determinate calculation.

The view which I had taken of the subject was more limited. Having obtained a law which expressed a particular class of these phenomena, I proceeded no further; but it was satisfactory to me to find, that as far as these extended, the expressions were identical, and that all the observed phenomena due to every variety of experiment were equally explicable on the one or the other hypothesis. It will not be necessary to enter further into the many beautiful effects which were obtained by the various arrangements of different galvanic conductors. I shall therefore proceed at once to describe the Experiment alluded to in the head of this article, which is intended, if not to prove, at least to show the high degree of probability, that all terrestrial magnetic phenomena are due only to electricity, and that magnetism, as a distinct quality, has no real existence.

Having, as stated in the preceding part of this paper, discovered that the magnetic power of an iron sphere resides only on its surface; having also shown that when we suppose the earth's magnetism to vanish, the fundamental laws of terrestrial magnetism are exhibited by this superficial action,—the resulting expressions being identical with those obtained by M. Biot for the earth; it occurred to me, by a very natural induction, that if I could distribute over the surface of an artificial globe a series of galvanic currents, in such a way that their tangential power should every where give a corresponding direction to the needle,—such a globe ought to exhibit, while under electrical induction, all the magnetic phenomena of the earth upon a needle freely suspended above it, the needle itself being neutralized from the earth's magnetism, so as to leave it wholly under the influence of this superficial action. This idea was put to the test of experiment as follows.

I procured a wooden globe sixteen inches in diameter, which was made hollow for the purpose of reducing its weight; and while still in the lathe, grooves were cut to represent an equator, and parallels of latitudes at every $4\frac{1}{2}^{\circ}$ each way from the equator to the poles; these grooves were about one

eighth of an inch deep and broad: and lastly, a groove of the same breadth, but of double the depth, was cut like a meridian, from pole to pole, half round. These grooves were for the purpose of laying in the wire, which was effected thus. The middle of a copper wire, nearly ninety feet long and one-tenth of an inch in diameter, was applied to the equatorial groove, so as to meet in the transverse meridian; it was then turned down that groove, one end towards one pole, the other towards the other pole, as far as the first parallel; it was then made to pass round this parallel, returned again along the meridian to the next parallel, then passed round this again, and so on till the wire was thus led in continuation from pole to pole.

The length of wire still remaining at each pole was bound with varnished silk, to prevent contact, and then returned from each pole along the meridian groove to the equator: at this point, each wire being fastened down with small staples, the two wires for the remaining five feet were bound together to near their common extremity, where they opened, to form two points for connecting the poles of a powerful galvanic battery. When this connection was made, the wire became of course an electric conductor, and the whole surface of the globe was put into a state of transient magnetic induction; and consequently, agreeably to the laws of action above described, a neutralized needle freely suspended above such a globe, would arrange itself in a plane passing from pole to pole through the centre, and take different angles of inclination according to its situation between the equator and either pole.

In order to render the experiment more strongly representative of the actual state of the earth, the globe in the state above described was covered by the gores of a common globe, which were laid on so as to bring the poles of this wire arrangement into the situation of the earth's magnetic poles, according to the best observations we have for this determination. I therefore placed them according to the mean results of Sir Edward Parry and Captain Foster in latitude 72° N. and 72° S., and on the meridian corresponding with longitude 76° W., by which means the magnetic and true equators cut one another in about 14° E. and 166° W. longitude.

The globe being thus completed*, a delicate needle must be suspended above

* This globe was constructed in 1824, and exhibited by Dr. Birkbeck, March 26th of that year, at the London Institution.

it, neutralized from the effect of the earth's magnetism, according to the principle I employed in my observations on the daily variation, and described in the Philosophical Transactions for 1823; by which means it will become entirely under the superficial galvanic arrangement just described. Conceive now the globe to be placed so as to bring London into the zenith; then, the two ends of the conducting wire being connected with the poles of a powerful battery*, it will be seen immediately that the needle, which was before indifferent to any direction, will have its north end depressed about 70°, as nearly as the eye can judge, which is the actual dip in London; it will also be directed towards the magnetic poles of this globe, thereby also showing a variation of about 24° or 25° to W., as is also the case in London. If now we turn the globe about on its support, so as to bring to the zenith places equally distant with England from the magnetic pole, we shall find the dip remains the same; but the variation will continually change, becoming first zero, and then gradually increasing to the eastward as happens on the earth. If again we turn the globe so as to make the pole approach the zenith, the dip will increase, till at the pole itself the needle will become perfectly vertical. Making now this pole recede, the dip will decrease, till at the equator it vanishes, the needle becoming horizontal. Continuing the motion, and approaching the south pole, the south end of the needle will be found to dip, increasing continually from the equator to the pole, where it becomes again vertical, but reversed as regards its verticality at the north pole.

Nothing can be expected nor desired to represent more exactly on so small a scale all the phenomena of terrestrial magnetism, than does this artificial globe: besides, we know from the mathematical laws of action which have been referred to, that it is not merely an exhibition of effects, but that if we could increase our currents indefinitely, every circumstance of dip and direction would admit of actual and accurate computation.

I may therefore, I trust, be allowed to say, that I have proved the existence of a force competent to produce all the phenomena of terrestrial magnetism, without the aid of any body usually called magnetic, except perhaps it may be

* My battery consisted of twenty zinc and twenty copper plates, ten inches square: but the experiment may be shown satisfactorily by Sturgeon's circular battery with about two feet of copper and one of zinc.

said the small needle employed for exhibiting the effects; and even this, but for its necessary minuteness, might be replaced by a galvanic needle on M. Ampere's construction. At the same time, I must beg again to observe, that it follows from the laws obtained by M. Biot, that no position of a single magnet, nor the arrangement of any numbers of such bodies within the globe, could by possibility exhibit the same phenomena, particularly as relates to the intensity of the needle.

I am quite aware that, after all, this does not amount to a demonstration that the magnetic phenomena of the earth are produced only by electricity; yet seeing as we do, in every operation of natural effects with which we are acquainted, that the agents employed are not more numerous than necessary, it will perhaps be admitted that I have at least shown the high probability that all terrestrial magnetic phenomena are due to some particular modification of electricity, and that magnetism, as a distinct quality, has no real existence in nature.

It is true, that as far as the discovery of Mr. Oersted goes, we have no idea how such a system of currents can have existence on the earth, because, to produce them, we have been obliged to employ a particular arrangement of metals, acids, and conductors; but, fortunately, a subsequent step, not less important than the former, was made by Professor Seebeck of Berlin, who discovered that the mere application of heat to a circuit composed of two metals, was competent to produce the same development of galvanic and magnetic effects as those above described; and there can be no doubt, that if the conducting wire of the globe I have described, were removed, and each parallel made complete in two metals, that all the phenomena it now exhibits by aid of the galvanic battery might be represented by the application of heat only.

The effect of heat is so obvious in the production of magnetic developments, that I have seen a rectangular circuit on Professor Seebeck's principle, constructed by Messrs. Watkins and Hill, which, by the momentary application of a spirit lamp, became sufficiently magnetic to deflect the needle several degrees, and even the minute change of temperature that could be produced by contact with the hand, exhibited a sensible effect: this circuit, however, was formed of two metals as proposed by Professor Seebeck; but Mr. Sturgeon of

Woolwich has been enabled to produce a similar effect with a rectangle of bismuth only.

This important discovery of M. SEEBECK brings us therefore, as it were, a step nearer to our object, by referring us to the sun as the great agent of all these phenomena; indeed but one link seems wanted to connect together a chain of highly interesting phenomena, and thereby to reduce to simple and intelligent principles what has hitherto been considered amongst the most mysterious laws of nature.

P.S. I have not in the above article made any reference to the irregularity of the magnetic lines on the earth. I have spoken of the law as deduced by M. BIOT, as if it answered accurately all the conditions required: it is however very well known that there are irregularities which it will not reach, and much credit is due to Professor HANSTEEN for the talent and industry he has applied in the collection of results, and the reduction of them to principles of calculation. These discrepancies however are by no means opposed to the foregoing view of the subject, but are, on the contrary, rather favourable to it; for if, as is implied in the preceding remark, the development of terrestrial magnetic phenomena be due to the transmission of caloric and inequality of temperature, we ought to expect the same kind of irregularities in this action as we know to exist in the temperature and climate of places situated geographically the same.

VII. *On the Equilibrium of Fluids, and the Figure of a Homogeneous Planet in a Fluid State.* By James Ivory, *A.M. F.R.S. Instit. Reg. Sc. Paris. Corresp. et Reg. Sc. Gottin. Corresp.*

Read January 13 and 20, 1831.

I. *Equilibrium of Fluids.*

1. THE nature of the ultimate particles of a fluid, and the peculiar manner of their mutual connection, are entirely unknown to us. We conceive that they obey the same mechanical laws to which all matter is subject. Experience shows that the particles of a fluid move freely among one another, yielding to the least pressure in any direction; and this is the most general property of such bodies that has yet been discovered. The perfect mobility of their particles must therefore, in the present state of our knowledge, be considered as constituting the definition of fluid bodies, and as the foundation of all our reasoning concerning them. We here confine our attention to a fluid in equilibrium, or at rest, in which state every particle is pressed equally on all sides. It is evident that the mobility of the particles among one another, and their readiness to obey any new impulse, is nowise impeded by the magnitude of their mutual pressure, since this acts at every point with the same intensity in all directions.

If we set aside the effect of gravity, and of all accelerating forces, it follows, from the definition, that the pressure will be equal in all parts of a continuous fluid at rest. In this state we must conceive that the particles are equally distant, and arranged similarly about every interior point. Their mutual distance, it is natural to think, must be connected with the magnitude of pressure; so that when they are more pressed, they will approach one another, and the volume will be diminished; and, when they are less pressed, they will recede from one another, and the volume will be enlarged. Accordingly it is found that no fluid is perfectly incompressible. But in some, such as water

and other liquids, a very great external force must be applied to produce an almost imperceptible variation of bulk; while in others, such as air and the gases, very notable changes of volume are caused by moderate compression. In the investigation of the properties of the first sort of fluids, to which our attention is here exclusively directed, we shall throw out of view the very small degree of compressibility they possess, and shall suppose them to retain the same bulk whatever changes of figure or pressure they may undergo.

In a fluid in equilibrium, the action of the accelerating forces that urge the particles must be counterbalanced by the pressure propagated through the mass: to find the relation between these opposite forces must therefore be the first object of research.

2. Assuming three planes intersecting at right angles which, by the co-ordinates drawn to them, ascertain the position of the particles of the fluid, we shall suppose two points or particles (x, y, z) and $(x+\delta x,\ y+\delta y,\ z+\delta z)$ at the infinitely small distance δs from one another; and we shall put ω for the small base of an upright cylinder or prism of the fluid placed between the two points, and having δs for its length: then the density of the fluid being invariable and represented by unit, and the quantity of matter of the cylinder or prism being denoted by dm, we shall have

$$dm = \omega \times \delta s.$$

Let all the accelerating forces which act upon the particle (x, y, z) be reduced to the directions of the coordinates; and put X, Y, Z for the sums of the reduced forces respectively parallel to x, y, z; then because $\frac{\delta x}{\delta s}, \frac{\delta y}{\delta s}, \frac{\delta z}{\delta s}$ are the cosines of the angles which the line δs makes with x, y, z, the partial forces urging the particle in the direction of δs, will be $X\frac{\delta x}{\delta s}$, $Y\frac{\delta y}{\delta s}$, $Z\frac{\delta z}{\delta s}$, and, if we put

$$f = X\frac{\delta x}{\delta s} + Y\frac{\delta y}{\delta s} + Z\frac{\delta z}{\delta s},$$

the whole accelerating force urging the particle (x, y, z) in the direction of δs, will be equal to f. Multiply now by the equal quantities dm and $\omega\delta s$, and the result will be

$$f\,dm = \omega\,(X\delta x + Y\delta y + Z\delta z).$$

As the quantities δx, δy, δz, δs, ω may be assumed as small as we please, the force f may be considered as retaining the same value for all the particles of the cylinder or prism; and therefore $f\,dm$ is the motive force of the cylinder or prism, or the effort it makes to move in the direction of δs from the point (x, y, z) to the point $(x + \delta x, y + \delta y, z + \delta z)$.

Let p represent the hydrostatic pressure of the fluid at the point (x, y, z). This term is used to denote the pressure relatively to the surface pressed: it is the whole pressure any surface sustains divided by the extent of surface; or it is the actual pressure reduced to the unit of surface. The hydrostatic pressure is obviously variable in the different parts of a fluid, the particles of which are urged by accelerating forces; and as it can vary only when its point of action is changed, it must be a function of the coordinates of that point. The whole pressure upon the end of the cylinder or prism at the point (x, y, z) will be equal to $p \times \omega$; for we may suppose that p undergoes no change in the small extent of the surface ω: and, in like manner, the whole pressure upon the opposite end will be equal to $(p + \delta p) \times \omega$. As the pressures upon the two ends act against one another, their effect to move the cylinder or prism in the direction of δs from the point $(x + \delta x, y + \delta y, z + \delta z)$ to the point (x, y, z) will be equal to $\delta p \times \omega$; and this force, on the supposition that the particles of the fluid are at rest, must be equal to $f\,dm$, the directly opposite effect caused by the accelerating forces. We therefore have this equation for expressing the non-effect of the equal and opposite forces, viz.

$$\delta p \times \omega + f\,dm = 0:$$

and, if we substitute the value of $f\,dm$ found before, we shall get

$$\delta p + \mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z = 0. \qquad (1)$$

This equation must take place at every point of the mass of fluid without any relation being supposed between the variations δx, δy, δz; which condition will not be fulfilled unless p be a function of the three independent variables x, y, z. We therefore have

$$\delta p = \frac{dp}{dx}\delta x + \frac{dp}{dy}\delta y + \frac{dp}{dz}\delta z:$$

and, if we substitute this value of δp in the formula (1), the independence of the variations will require these three separate equations,

$$\frac{dp}{dx} = -X$$

$$\frac{dp}{dy} = -Y$$

$$\frac{dp}{dz} = -Z.$$

From this it appears that the algebraic expressions of the forces are not entirely arbitrary; for they must be equal to the partial differential coefficients of a function of three independent variables. By differentiating we shall readily obtain the following equations which do not contain the function p, viz.

$$\frac{dX}{dy} = \frac{dY}{dx}, \quad \frac{dX}{dz} = \frac{dZ}{dx}, \quad \frac{dY}{dz} = \frac{dZ}{dy}.$$

Unless the forces possess these properties, which are the well-known conditions of integrability, the equation (1) will not hold in all parts of the mass of fluid, and the equilibrium will be impossible. But in the physical questions that actually occur, the forces of nature being either attractions or repulsions directed to fixt centres, and proportional to certain functions of the distances from those centres, they necessarily fulfil the conditions of integrability.

The whole of what has been said is succinctly expressed by the two following equations,

$$\left.\begin{aligned} \varphi &= \int (X\,dx + Y\,dy + Z\,dz), \\ p &= C - \varphi, \end{aligned}\right\} \qquad (2)$$

Here φ represents a function of three independent variables x, y, z without any arbitrary quantity; the constant C required by the integration is necessary only in the expression of p.

3. The hydrostatic pressure at every point of the mass of fluid in equilibrium, is expressed by the second of the equations (2), viz.

$$p = C - \varphi.$$

But at all those parts of the outer surface of the fluid which are unconfined and entirely at liberty, there is no pressure; wherefore we have, for the equation of all such surfaces,

$$\varphi = C.$$

It may be proper to remark, that although this equation is universally true, yet it is no new or independent condition of the equilibrium; it is merely an inference from the general expression of the hydrostatic pressure.

If we assume two points (x, y, z) and $(x+dx,\ y+dy,\ z+dz)$ indefinitely near one another in a part of the outer surface at liberty, we shall have, in consequence of the foregoing equation,

$$\frac{d\Phi}{dx}dx+\frac{d\Phi}{dy}dy+\frac{d\Phi}{dz}dz=0;$$

or, which is the same thing,

$$\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz=0;$$

and if ds represent the distance of the two points, we obtain

$$\mathrm{X}\frac{dx}{ds}+\mathrm{Y}\frac{dy}{ds}+\mathrm{Z}\frac{dz}{ds}=0.$$

Now $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the cosines of the angles which the directions of the forces make with the line ds; wherefore the expression on the left side of the foregoing formula is the sum of the partial forces which act in the direction of ds; and as this sum is equal to zero in all positions of the line ds round the point (x, y, z), the resultant of the forces produces no effect in the plane touching the surface, and consequently its whole action is perpendicular to that plane. The nature of the case requires further, that the same resultant be directed towards the surface of the fluid.

What has been deduced from the algebraic expressions is evident in another view. For, could we suppose that the resultant of the forces is not at every point perpendicular to the surface at liberty, it might be resolved into two partial forces, one acting in the tangent plane, and the other perpendicular to that plane; and as the first force is opposed by no obstacle, it would cause the particles to move, which is contrary to the equilibrium.

If we suppose that p is constant in the general formula of the hydrostatic pressure, we shall have an equation,

$$\varphi=\mathrm{C}-p,$$

which is exactly similar to that of the surface at liberty, and which will determine an interior surface at every point of which there is the same intensity of

pressure. By differentiating the equation of the interior surface, we obtain

$$X\,dx + Y\,dy + Z\,dz = 0;$$

from which we deduce, by the like reasoning as before, that such surfaces are perpendicular to the resultant of the accelerating forces urging the particles contained in them. The interior surfaces in question were named level surfaces by CLAIRAUT; and they are distinguished by the two properties of being equally pressed at all their points, and of cutting the resultant of the forces at right angles. They spread through the mass, and ultimately coincide with those parts of the outer surface which are at liberty. It may be observed, that what essentially constitutes a level surface is its equation, which must differ from the equation of the outer surface at liberty in no respect, except that the constant $C - p$ takes the place of the constant C; for we shall afterwards find that, in some cases of the equilibrium of a fluid, the two properties of being equally pressed, and of cutting the resultant of the forces at right angles, belong to more sets of interior surfaces than one.

4. In what goes before, we have supposed that the density is constant, but it is easy to extend the investigation to heterogeneous fluids. Let ϱ be put for the function of the co-ordinates which expresses the variable density; then admitting that ϱ has the same value at every point of the small elementary cylinder or prism, we shall have

$$dm = \varrho\,\omega\,\delta s;$$

but, f being the whole accelerating force, urging every particle of dm in the direction of δs, we have

$$f = X\frac{\delta x}{\delta s} + Y\frac{\delta y}{\delta s} + Z\frac{\delta z}{\delta s};$$

wherefore,

$$f\,dm = \varrho\,\omega\,(X\delta x + Y\delta y + Z\delta z).$$

The equation expressing that the action of the accelerating forces is equal and opposite to the variation of pressure, is the same as before, viz.

$$\omega \times \delta p + f\,dm = 0;$$

and by substituting the value of $f\,dm$, we deduce

$$\delta p + \varrho\,(X\delta x + Y\delta y + Z\delta z) = 0. \qquad (3)$$

This equation must hold at every point of the mass of fluid without any relation being supposed between the variations, wherefore p must be a function of three independent variables; and in consequence the foregoing equation implies the three separate equations following, viz.

$$\frac{dp}{dx} = -\varrho X, \quad \frac{dp}{dy} = -\varrho Y, \quad \frac{dp}{dz} = -\varrho Z.$$

It now appears that the conditions of integrability must be fulfilled, viz.

$$\frac{d.\varrho X}{dy} = \frac{d.\varrho Y}{dx}, \quad \frac{d.\varrho X}{dz} = \frac{d.\varrho Z}{dx}, \quad \frac{d.\varrho Y}{dz} = \frac{d.\varrho Z}{dy};$$

and unless the forces possess the properties expressed by these equations, the equilibrium will be impossible.

Without pursuing the investigation in all its generality, we shall confine our attention to the case in which

$$X\,dx + Y\,dy + Z\,dz,$$

is an exact differential; a supposition that comprehends all the applications of the theory. If we represent the integral of the differential by φ, so that

$$d\varphi = X\,dx + Y\,dy + Z\,dz;$$

and convert the variations of equation (3) into differentials, we shall obtain

$$dp + \varrho\,d\varphi = 0;$$

and hence

$$p = C - \int \varrho\,d\varphi. \qquad (4)$$

From this we deduce the equation of those parts of the outer surface which are at liberty, by making $p = 0$; and that of a level surface, by assigning to p some constant value. And if we differentiate the same equation (4) on the supposition that p is invariable, we shall get

$$\varrho\,d\varphi = \varrho\,(X\,dx + Y\,dy + Z\,dz) = 0,$$

which differential equation is common to the outer surface at liberty, and to all the interior level surfaces; and from which we deduce by the like reasoning as before, that all such surfaces are perpendicular to the resultant of the accelerating forces urging the particles contained in them.

The quantity under the sign of integration in the formula,

$$p = C - \int \varrho\,d\varphi,$$

must be an exact differential, for p must be a function of the co-ordinates; which condition will not be fulfilled unless ϱ be a function of φ. Thus both the pressure p and the density ϱ are functions of the same quantity φ, and they are both invariable where φ is constant. The density is therefore the same at all the points of any level surface. If we conceive a heterogeneous fluid in equilibrium to be divided into thin strata by level surfaces infinitely near one another, the density will be the same throughout every stratum, but it will vary from one stratum to another.

5. We have now placed before the reader the general points of the theory of the equilibrium of fluids. What has been said comprehends all that can be determined when a fluid is conceived to extend indefinitely; but in applying the theory to limited masses, it is necessary besides, that the pressures propagated through the interior parts either be supported or mutually balance one another.

In treating of the equilibrium of fluids, another mode of investigation is sometimes employed, which it would be improper to pass by without notice, as it is useful on many occasions to fix the imagination, although it leads to no new results. We allude to the narrow canals supposed to traverse the mass in various ways, of which so much use has been made by CLAIRAUT and other authors.

Let two points $(x^\circ, y^\circ, z^\circ)$ and (x', y', z') be assumed in the interior of a mass of fluid in equilibrium, and conceive an infinitely narrow canal of any figure to pass between them; we may suppose that the whole fluid, except the portion within the canal, becomes solid without any change taking place in the position of the particles, or in their mutual action upon one another; for, as this supposition makes no alteration of the forces urging the particles contained within the canal, these particles will remain at rest after the solidification as they were at first. Suppose that the canal is divided into infinitely small parts by sections perpendicular to its sides; at any point (x, y, z) let ω be the section; δs the infinitely small part of the length of the canal; dm the quantity of matter in the length δs, that is, the product of the volume and the density, or $\varrho \times \omega \times \delta s$; and f the sum of all the partial forces that urge the particles of dm in the direction of the canal; then, the motive force of dm, or its effort to move, will be equal to $f\,dm$. Further, p being the hydrostatic

pressure at the point (x, y, z), the like pressure at the distance of δs will be $p + \delta p$; therefore the opposing pressures which act upon the two ends of the part of the canal in the length δs, will be $p \times \omega$ and $(p + \delta p) \times \omega$; and $\delta p \times \omega$ will be the effective pressure which pushes dm towards the point (x, y, z). Because every part of the canal is supposed at rest, the tendencies of dm to move in opposite directions must be equal, and we shall have this equation,

$$\delta p \times \omega + f\,dm = 0;$$

consequently,

$$\delta p + \frac{f\,dm}{\omega} = 0;$$

and by taking the sum of the similar quantities in all the parts of the canal, we obtain

$$\int \delta p + \int \frac{f\,dm}{\omega} = 0.$$

But p being a function of three independent variables, the sum of its variations, supposing the flowing quantities to follow any arbitrary law of increase or decrease, is equal to the difference of p' and p°, the final and initial values of the function; wherefore we have

$$p' - p^\circ + \int \frac{f\,dm}{\omega} = 0.$$

Now $f\,dm$, that is the quantity of matter multiplied by the accelerating force, is the impulse or pressure in the direction of the canal caused by all the forces urging dm; and as this pressure is exerted on the surface ω, $\frac{f\,dm}{\omega}$ is the same pressure reduced to the unit of surface. Therefore, whatever be the figure of the canal, it follows from the foregoing investigation, that the difference of the pressures at its two extremities is equal to the sum of the impulses of all the contained molecules of fluid, every impulse being reduced to the direction of the canal and to the unit of surface.

If the extremities of the canal be both in the parts of the outer surface which are at liberty, the pressures p' and p° will be both evanescent, and there will be no effort of the fluid either way, and no tendency to run out at one end. Further, if a canal be continued through the fluid till it return into itself, the

initial and final pressures being the same, the impulses of the molecules in the whole circuit will balance one another. But in this case, the reasoning we have employed will not be exact, unless p, the algebraic expression of the pressure, be such a function as admits of only one value for any three given co-ordinates; a restriction however, which, in every point of view, seems indispensable.

6. The whole theory, it will readily appear from the foregoing investigations, is built on the assumption, That the hydrostatic pressure at every point of the fluid is the same function of the co-ordinates of the point. The accelerating forces are represented by the partial differential coefficients of the pressure; and therefore they are likewise the same functions of the co-ordinates of their point of action in every part of the mass. The whole reasoning rests on these fundamental points; and if the state of a fluid were such that they are not verified, the equations for determining the required figure could not be formed, and the equilibrium would be impossible. As the hydrostatic pressure is known only by means of the given accelerating forces, it seems most suitable to employ the properties of the latter in laying down what is required for the equilibrium of a mass of fluid. It is necessary, and it is sufficient for the equilibrium of a homogeneous fluid, first, that the accelerating forces acting in the directions of the co-ordinates be, in every part of the mass, the same functions of the co-ordinates; and, secondly, that these functions possess the conditions of integrability. When these two conditions are both fulfilled, the determination of the figure of equilibrium is reduced to a question purely mathematical. For we can form the equation (1) which makes the accelerating forces balance the variation of pressure; and, by integrating this equation, we obtain the hydrostatic pressure, from which is deduced the equation of all those points at which there is no pressure, or in other words, the equation of all those parts of the outer surface which are at liberty. Nothing more is required for securing the permanence of the figure of the fluid, except that the pressures propagated through the mass be either supported or mutually balance one another.

The conditions for the equilibrium of a homogeneous fluid, as they are here laid down, do not enable us in all cases to form immediately the equation of the figure of equilibrium. If the particles attract or repel one another, the accelerating forces will, for the most part, vary as the fluid changes its form,

and they may not be at every point the same functions of the co-ordinates in all the figures, of which it is susceptible; but, notwithstanding the equilibrium may still be possible, because this indispensable condition may be fulfilled when figures of a certain class are induced on the mass. In such cases, the determination of the equilibrium necessarily requires two distinct researches; of which one is to find out what are the particular figures into which the mass must be moulded, so as to make the accelerating forces at every point the same functions of the co-ordinates. After these figures have been found, we can apply to them the equations expressing the conditions of equilibrium, and accomplish the mathematical solution of the problem. But if it shall appear that no figure whatever capable of fulfilling both the conditions laid down above can be induced on the fluid, the equilibrium will be absolutely impossible.

In the usual exposition of this theory, the equilibrium is made to depend on conditions that do not exactly coincide with those at which we have arrived. According to CLAIRAUT and all other authors who have written on this subject, it is necessary, and it is sufficient, for the equilibrium of a homogeneous fluid, first, that the expressions of the accelerating forces possess the criterion of integrability; secondly, that the resultant of the forces in action at all the parts of the outer surface which are at liberty, be directed perpendicularly towards these surfaces. We may throw out of view what regards the criterion of integrability, about which there is no difference of opinion, and which in reality is always fulfilled by the forces that occur in physical researches. The perpendicularity of the forces to the outer surface is a property of the differential equation of that surface, and will necessarily take place whenever it is possible to form that equation. Nothing more is required for forming the equation mentioned, than that the accelerating forces at every point of it be expressed by the same functions of the co-ordinates of the point.* It follows

* The forces are perpendicular to every surface in which the pressure is constant. The outer surfaces are those at every point of which there is no pressure. In all the questions that have occurred, the forces at the outer surface of the fluid are the same functions of the co-ordinates of the point, whatever geometrical figure the fluid is supposed to assume; and on this account the equation of the outer surface can be formed without reference to any particular class of figures. But this is not sufficient; for, according to the fundamental assumption laid down by CLAIRAUT himself, the theory of equilibrium cannot be applied, unless the forces be the same functions of the co-ordinates of their point of action in every part of the mass.

therefore, that the difference between the conditions of equilibrium hitherto universally adopted, and those laid down above, amounts to this: according to the former it is required that the expressions of the accelerating forces be the same functions of the co-ordinates at every point of the outer surface, this being all that is necessary for forming the differential equation of that surface; according to the latter, the forces will not balance the pressure, and the laws of equilibrium will not be fulfilled unless the forces be the same functions of the co-ordinates at every point whether situated in the outer surface, or in the interior part of the mass.

If a homogeneous fluid, of which the particles are urged by accelerating forces be in equilibrium, all that is required by CLAIRAUT's theory will undoubtedly be fulfilled; but the converse of this cannot be affirmed. It is no where proved generally by unexceptionable arguments, and indeed no proof can possibly be given, that the forces in the interior parts of the fluid will balance the pressure, merely because the resultant of the forces in action at the outer surface is perpendicular to that surface. All the attempts that have been made to demonstrate this point, tacitly assume that the expression of the forces is the same at the surface and in all the interior parts; which is not universally true.

In a very extensive class of problems the difference between the two ways of laying down the conditions of equilibrium disappears. This will happen when the accelerating forces are independent of the figure of the fluid, as will be the case if the particles exert no action on one another by attraction or repulsion. In such problems the forces impressed upon every particle, whatever be its situation, and whatever be the figure of the fluid, are by the hypothesis, the same given functions of the co-ordinates. The figure of equilibrium will be the same whether, following CLAIRAUT, we obtain the equation of the outer surface by means of the forces in action at that surface, or, making use of the property that the pressure vanishes at all the points where the fluid is at liberty, we deduce the same equation from the pressure that prevails generally throughout the mass.

But CLAIRAUT's theory cannot be extended to the solution of other problems than those of which we have been speaking. In no other cases is it evident without inquiry that the proposed accelerating forces urging a particle, are, in

every part of the mass, the same functions of the coordinates of the particle; and unless this be verified, the theory of equilibrium cannot be applied. In a homogeneous planet in a fluid state, there are forces which prevail in the interior parts and vanish at the surface; and, as CLAIRAUT's theory notices no forces except those in action at the surface, it leaves out some of the causes tending to change the figure of the fluid, and therefore it cannot lead to an exact determination of the equilibrium.

II. *Application of the foregoing Theory to the Question of the Figure of the Planets.*

7. Having now explained the general theory of the equilibrium of fluids at sufficient length, I proceed to apply it to the question of the figure of the planets, in which it is required to determine the equilibrium of a fluid entirely at liberty, and unconfined by any obstacle or support. The problem is one of considerable difficulty. It is necessary to distribute the investigation under distinct heads. It would otherwise be impossible to preserve perspicuity and precision of ideas in an inquiry essentially different in different hypotheses. The equilibrium of a homogeneous fluid must occupy our attention before that of one having its density variable. For although it may at first appear that the latter problem is the more general, and includes the former, yet it will be found that the equilibrium of a fluid of variable density, depends upon that of a homogeneous fluid, and is deducible from it. And even with regard to homogeneous fluids, distinctions must be made, because what is required for the equilibrium varies with the nature of the accelerating forces. In this respect we distinguish these two general cases, of which we shall treat in two separate problems; First, when the accelerating forces depend only on the coordinates of their point of action, and are explicitly known when the coordinates are given; Secondly, when the accelerating forces depend not only upon the coordinates of the particle on which they act, but likewise upon the figure of the whole mass of fluid; as happens for the most part when the particles attract or repel one another.

Problem 1st.—To determine the equilibrium of a homogeneous mass of fluid which is entirely at liberty, when the accelerating forces are known functions of the coordinates of their point of action.

The equilibrium of a mass of fluid which is entirely at liberty, can depend only upon the action of such forces as tend to change the relative position of the particles with respect to one another. It is not affected by any motion common to all the particles, nor by any force which acts upon them all with the same intensity in the same direction; the effect of such motion, or of such force, being to displace the centre of gravity of the whole mass without altering the relative situation of the particles. In estimating the accelerating forces upon which the figure of equilibrium will depend, we must therefore begin with reducing the centre of gravity, if it be in motion or urged by any force, to a state of relative rest; which is accomplished by applying to every particle a force that would cause it to move with the same velocity as the centre of gravity, but in a contrary direction. In the investigation of this problem we may therefore suppose that the centre of gravity is at rest and undisturbed by the action of any accelerating force.

Suppose now that a mass of homogeneous fluid entirely at liberty, is in equilibrium, and conceive three planes intersecting at right angles in the centre of gravity of the mass, to which planes the particles of the fluid are to be referred by rectangular coordinates. Let x, y, z, represent the coordinates of a particle, and having resolved the accelerating forces acting upon it into other forces that have their directions parallel to the coordinates, put X, Y, Z, for the sums of the resolved parts respectively parallel to x, y, z, and tending to shorten these lines. According to the hypothesis of this problem, the forces X, Y, Z, depend only upon the coordinates of their point of action; and they are at every point the same functions of those coordinates. The equilibrium will therefore be impossible unless

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz$$

be an exact differential, this being necessary in order that the hydrostatic pressure be a function of three independent variables as the fundamental assumption of the theory demands. Let φ denote the integral, and p the

hydrostatic pressure at the point $(x, y, z,)$: the equations that determine the equilibrium will be these two *,

$$\left.\begin{aligned} \varphi &= \int (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz), \\ p &= \mathrm{C} - \varphi. \end{aligned}\right\},$$

If we make $p = 0$, we shall obtain the equation of the outer surface of the fluid, viz.

$$\varphi = \mathrm{C}.$$

The differential equation,

$$\frac{d\varphi}{dx}\,dx + \frac{d\varphi}{dy}\,dy + \frac{d\varphi}{dz}\,dz = 0,$$

or which is the same,

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = 0,$$

is common to the outer surface and to all the interior level surfaces at every point of which there is the same intensity of pressure; and it shows that the resultant of the accelerating forces is perpendicular to all such surfaces †.

The figure of the fluid being determined, it remains to inquire whether the equilibrium is secured. By varying the coordinates in the formula for p, we obtain

$$\delta p + \frac{d\varphi}{dx}\delta x + \frac{d\varphi}{dy}\delta y + \frac{d\varphi}{dz}\delta z = 0:$$

which equation proves that, if a particle be moved from its place a very little in any direction, the variation of the intensity of pressure is equal and opposite to the action of the accelerating forces. A particle has therefore no tendency to move from inequality of pressure. But we must not from this hastily conclude that there is no cause tending to change the figure of the fluid. For, as in the simple case of a fluid contained in a vessel, the equilibrium requires not only that the accelerating forces balance the inequality of pressure, but likewise that the total pressures tending outward at the boundaries of the mass, be supported by the sides of the vessel; so in the problem under consideration, there being no external support, the figure of the fluid will not be permanent

* Equation (2) § 2. † Equation (2) § 3.

unless the pressures propagated inward, which increase as any point sinks deeper below the surface, mutually compensate and destroy one another. Some further discussion is therefore necessary in order to prove that the equilibrium is completely established.

The function φ, in which we may suppose there is no constant quantity, can contain no term having the coordinates for divisors; for, were this the case, the pressure would be infinite at all those points where such coordinates are equal to zero. Let the terms of φ be arranged in homogeneous expressions of one, two, three, &c. dimensions; then

$$\begin{aligned}\varphi = &(A_1 x + A_2 y + A_3 z)\\ &+ (B_1 x^2 + B_2 y^2 + B_3 z^2 + B_4 x y + B_5 x z + B_6 y z)\\ &+ (D_1 x^3 + D_2 y^3 + D_3 z^3 + D_4 x^2 y + \&c.)\\ &+ \&c.\end{aligned}$$

Differentiate this expression, and after the operations put $x = 0$, $y = 0$, $z = 0$: then

$$\frac{d\varphi}{dx} = A_1, \quad \frac{d\varphi}{dy} = A_2, \quad \frac{d\varphi}{dz} = A_3.$$

But the differentials of φ are no other than the expressions of the accelerating forces acting on a particle; consequently A_1, A_2, A_3 are the forces in action at the origin of the coordinates, that is, at the centre of gravity of the mass. Wherefore, according to what was observed, we shall have

$$\begin{aligned}&A_1 = 0, \quad A_2 = 0, \quad A_3 = 0,\\ \varphi = &(B_1 x^2 + B_2 y^2 + B_3 z^2 + B_4 x y + B_5 x z + B_6 y z)\\ &+ (D_1 x^3 + D_2 y^3 + D_3 z^3 + D_4 x^2 y + \&c.)\\ &+ \&c.\end{aligned}$$

That the expression of φ must be of this form is required by the nature of the problem: for φ must be always positive, and it must increase continually from the centre of gravity to the surface of the fluid.

Let us now put

$$\begin{aligned}x &= r\cos\theta = r\xi,\\ y &= r\sin\theta\cos\psi = r\eta,\\ z &= r\sin\theta\sin\psi = r\zeta,\end{aligned}$$

then r will be the line drawn from the centre to the point (x, y, z); and the arcs θ and ψ determine the direction of r, θ being the angle between r and the axis of the coordinates parallel to x, and ψ the angle which the plane containing r and the same axis makes with the plane of x, y. By substituting, we get

$$\begin{aligned}\varphi = {} & r^2 (B_1 \xi^2 + B_2 \eta^2 + B_3 \zeta^2 + B_4 \xi\eta + B_5 \xi\zeta + B_6 \eta\zeta) \\ & + r^3 (D_1 \xi^3 + D_2 \eta^3 + D_3 \zeta^3 + D \xi^2 \eta + \&c.) \\ & + \&c.\end{aligned}$$

The symbols ξ, η, ζ, represent three rectangular coordinates of a point in the surface of a sphere having unit for its radius; and, in order to simplify, I shall write Q_2, Q_3, and generally Q_n, for homogeneous functions of ξ, η, ζ, of two, three, and n dimensions: then,

$$\varphi = r^2 Q_2 + r^3 Q_3 + r^4 Q_n + \&c.$$

For the sake of distinction, let R represent a line drawn from the centre of gravity to the surface of the fluid; and r a line drawn from the same centre to any interior point at which the pressure is p, the directions in which R and r are drawn being determined by the arcs θ and ψ: the equation of the fluid's surface, and the expression of p, will be as follows,

$$\begin{aligned} C &= R^2 Q_2 + R^3 Q_3 + R^4 Q_n + \&c. \\ p &= C - (r^2 Q_2 + r^3 Q_3 + r^4 Q_n + \&c.) \end{aligned}$$

By means of these equations a radius, R or r, will be known when the arcs θ and ψ which determine its direction are assumed; and in this manner we may find all the points of the outer surface, and of any interior level surface in which p has any assigned value less than C. All these surfaces will return into themselves and inclose a space: because in whatever direction we proceed from the centre of gravity to the surface, the function φ passes through every gradation of magnitude between zero and the maximum.

It is now easy to complete the demonstration of the equilibrium. A stratum of the fluid between the outer surface and any interior level surface will evidently be in equilibrium, if we suppose that the level surface maintains its figure, or rather, that there are no forces urging the particles contained within that surface: for, the upper part of the stratum cuts the resultant of the forces

at right angles, and the fluid presses perpendicularly and with the same intensity at every point of the lower surface which supports the stratum. What is here affirmed is true, however near the level surface be to the centre of gravity; and as the accelerating forces urging the particles within the surface decrease without limit in approaching that centre, they may finally be regarded as evanescent when the internal body of fluid is no more than a drop occupying the centre of gravity. Wherefore, by taking the radius of the level surface small enough, the inclosed fluid may be considered free from any accelerating forces, and subject only to the external pressures; and, these being perpendicular to the surface, and acting with the same intensity, the whole mass of fluid will be in equilibrium by the known laws of hydrostatics.

It may be proper to add that the mass of fluid has no tendency to turn upon an axis. For no motion of this kind can be produced by the pressures propagated inward from the surface, the directions of which pass through the centre of gravity. Neither can the accelerating forces urging the particles, cause any such motion, these being wholly employed in counteracting the inequality of pressure.

For the sake of illustrating the problem we have solved, we shall add one example, which is besides intimately connected with the principal subject of our research.

Example.—To determine the figure of equilibrium of a homogeneous mass of fluid entirely at liberty, the particles being supposed to attract one another with a force directly proportional to the distance at the same time that they are urged by a centrifugal force caused by rotation about an axis.

At first view the proposed problem may seem one in which the accelerating forces depend upon the figure of the fluid, since it is supposed that every particle is attracted by every other. But, in the particular law of attraction assumed, the force which urges any particle is directed to the centre of gravity of the whole mass of matter, and is proportional to the distance from that point *. The hypothesis of the problem is therefore equivalent to the supposition that the particles of the fluid are attracted to a fixt centre with a force proportional to the distance; so that the accelerating forces are independent of the figure of the fluid.

* Prin. Math. Lib. i. Prop. 88.

As the centre of gravity of a mass of fluid in equilibrium must be free from the action of any force, except what is common to all the particles; and as the attractions of the particles balance one another at that point; the centrifugal force must likewise be evanescent at the same point, and consequently the axis of rotation must pass through it. Let three planes intersecting at right angles, one being perpendicular to the axis of rotation, pass through the centre of gravity; and assuming any particle of the fluid, let r denote its distance from the same centre, and x, y, z its coordinates, z being parallel to the axis of rotation: further, let g represent the attractive force of the whole mass of fluid at the distance equal to unit from the centre of gravity; and f the centrifugal force (that is, its proportion to g) at the distance equal to unit from the axis of rotation: then $g r$ will be the central attraction urging the particle, and $g x$, $g y$, $g z$, will be the resolved parts of the same force in the directions of the coordinates: also, $\sqrt{x^2+y^2}$ will be the distance of the particle from the axis of rotation; $-f\sqrt{x^2+y^2}$, the whole centrifugal force estimated as tending to shorten the coordinates; and $-fx$, $-fy$, the resolved parts of the same force, parallel to x and y: collecting, now, the partial forces which urge the particle in the respective directions of the coordinates, we shall find,

$$X = (g-f)\,x, \qquad Y = (g-f)\,y, \qquad Z = g\,z.$$

The equations of equilibrium will, therefore, be

$$\varphi = \int (X\,dx + Y\,dy + Z\,dz) = \tfrac{1}{2}\{(g-f)\,(x^2+y^2) + g\,z^2\},$$
$$p = C - \tfrac{1}{2}\{(g-f)\,(x^2+y^2) + g\,z^2\}$$

The equation of the surface of the fluid will be found by making $p = 0$, viz.

$$C = \tfrac{1}{2}(g-f)\,(x^2+y^2) + \tfrac{1}{2}g\,z^2.$$

And, if we put $e^2 = \frac{f}{g}$, the same equation may be thus written,

$$a^2 = x^2 + y^2 + \frac{z^2}{1-e^2},$$

which belongs to an elliptical spheroid of revolution having the equatorial semidiameter equal to a, and the polar semi-axis to $a\sqrt{1-e^2}$.

8. The order of discussion that has been laid down now brings us to the

more difficult part of this research, when the accelerating forces urging the particles of the fluid, depend upon the very figure of equilibrium which is to be investigated. This must happen in fluids consisting of particles that mutually attract one another, if the attractive force acting upon a particle vary with the figure of the attracting matter. In this division of our subject, the law of attraction that prevails in nature being in reality the only one which it is of much importance to consider, will chiefly engage attention.

Problem 2nd.—To determine the equilibrium of a homogeneous fluid entirely at liberty, the particles attracting one another with a force inversely proportional to the square of the distance, at the same time that they are urged by a centrifugal force caused by rotation about an axis.

The fluid being supposed in equilibrium, the axis of rotation must pass through the centre of gravity of the mass. For, abstracting from any motion or force common to all the particles, that centre may be considered at rest and free from the action of any accelerating force; and, as the attractive forces balance one another at that point, the centrifugal force must likewise vanish at the same point.

Conceive three planes intersecting at right angles in the centre of gravity of the mass, one of them being perpendicular to the axis of rotation: let x, y, z represent the coordinates of a particle in the surface of the fluid, x being parallel to the same axis; and put V for the sum of the quotients of all the molecules of the mass divided by their respective distances from the particle: then the attractive forces urging the particle inward in the directions of x, y, z, will be respectively equal to

$$-\frac{d\mathrm{V}}{dx}, \quad -\frac{d\mathrm{V}}{dy}, \quad -\frac{d\mathrm{V}}{dz}.$$

Further, if f denote the centrifugal force at the distance unit from the axis of rotation, the action of the same force at the distance $\sqrt{y^2+z^2}$ from the same axis will be $f\sqrt{y^2+z^2}$; and the resolved parts of this force urging the particle to move in the prolongations of y and z, will be fy and fz. Wherefore the total forces parallel to x, y, z, and tending to shorten these lines, are respectively,

$$-\frac{dV}{dx},\ -\left(\frac{dV}{dy}+fy\right),\ -\left(\frac{dV}{dz}+fz\right):$$

and the condition that the resultant of these forces is perpendicular to the surface of the fluid is expressed by this differential equation,

$$\frac{dV}{dx}dx+\frac{dV}{dy}dy+\frac{dV}{dz}dz+f(y\,dy+z\,dz)=0;$$

and the integral, viz.

$$C=V+\frac{f}{2}(y^2+z^2),$$

is the equation of the surface of the fluid in equilibrium. This is incontestably the true equation of the surface in equilibrium, since all the forces in action at that surface have been taken into account.

Using x, y, z to represent generally the co-ordinates of any particle of the mass, and the symbol V, to denote the function of x, y, z, which is equal to the sum of the quotients of all the molecules of the mass of fluid divided by their respective distances from the particle, it will be convenient to have some means of pointing out whether V belongs to a point in the surface, or to one differently situated. For this purpose we shall put $r=\sqrt{x^2+y^2+z^2}$ for the distance from the centre of gravity, and shall write V (r) for the value of V relatively to a point within the mass; and we shall suppose that r becomes R at the upper surface, so that V (R) will denote the value of V for a point in that surface. According to this notation, the foregoing equation of the surface of the fluid in equilibrium, will be thus written,

$$C=V(R)+\frac{f}{2}(y^2+z^2). \qquad (1)$$

The attraction of the whole mass and the centrifugal force, which are the only forces that urge a particle in the upper surface, likewise act upon every particle in the interior parts of the fluid. It will contribute to perspicuity if to these forces we give the name of the *principal forces*, in order to distinguish them from any other forces which an attentive examination may enable us to detect. Assuming any molecule in the interior parts, r being its distance from the centre of gravity, and x, y, z its coordinates, we have only to proceed as before, writing V (r) for V, in order to find the resolved parts of the principal

forces which urge the molecule inward in the respective directions of x, y, z, viz.

$$-\frac{d\,\mathrm{V}(r)}{dx}, \qquad -\left(\frac{d\,\mathrm{V}(r)}{dy}+fy\right), \qquad -\left(\frac{d\,\mathrm{V}(r)}{dz}+fz\right):$$

and if these forces be multiplied, each by the variation of its direction, the sum of the products will be the variation of the intensity of pressure, which is equal and opposite to their action, according to equation (1) of the general theory; thus, we have,

$$\delta p-\frac{d.\,\mathrm{V}(r)}{dx}\delta x-\frac{d.\,\mathrm{V}(r)}{dy}\delta y-\frac{d.\,\mathrm{V}(r)}{dz}\delta z-f(y\,\delta y+z\,\delta z)=0; \qquad (2)$$

and, as this equation is true at every point of the mass, we further obtain

$$p=\mathrm{V}(r)+\frac{f}{2}(y^2+z^2)-\mathrm{C}, \qquad (3)$$

the constant being the same as in the equation (1) of the upper surface, because the two equations must coincide when the interior molecule ascends to the surface. It must be observed that p represents the intensity of pressure caused by the principal forces alone, and not the whole pressure upon the molecule, if besides these forces there exist other causes of pressure in the interior parts.

From the nature of the function V or V (r), it has its maximum at the centre of gravity of the mass, or when $r=0$; for at that point we have the equations

$$\frac{d.\,\mathrm{V}(r)}{dx}=0, \qquad \frac{d.\,\mathrm{V}(r)}{dy}=0, \qquad \frac{d.\,\mathrm{V}(r)}{dz}=0,$$

because the attractive forces balance one another. While r, without any change in its direction, increases to be equal to R, V (r) continually decreases. In whatever direction the radius R be drawn to the surface, there is always a point in it, the coordinates of which will satisfy equation (3), supposing that p has any assigned value less than the maximum which takes place at the centre of gravity. All the points in which p has the same given value will form an interior surface, returning into itself and pressed with equal intensity by the action of the principal forces upon the exterior fluid. Such interior surfaces are likewise perpendicular to the resultant of the principal forces

urging the particles contained in them, as will readily be proved by differentiating equation (3), making p constant.

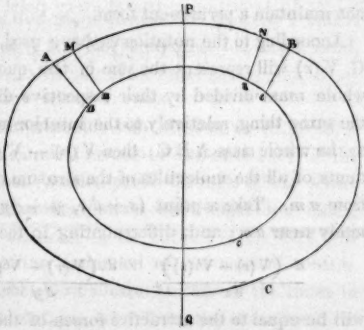
In order to place what has been said in the clearest light, let A B C represent the mass of fluid, the surface being determined by the equation,

$$C = V(R) + \frac{f}{2}(y^2 + z^2);$$

and suppose that $a\,b\,c$ is an interior surface, obtained by making p constant in the equation,

$$p = V(r) + \frac{f}{2}(y^2 + z^2) - C:$$

then, if the narrow canal A $a\,m$ M stand upon the molecule $a\,m$ of the interior surface, and extend to the upper surface of the fluid, the intensity of pressure upon $a\,m$, or the given quantity p, will be equal to the sum of all the impulses caused by the action of the principal forces upon the molecules contained in the canal, every impulse being reduced to the direction of the canal and to the unit of surface. The same thing is true of any other molecule in the same surface upon which there stands a similar canal B $b\,n$ N.



If we attend to the conditions of equilibrium required by the general theory, it will readily appear that the equilibrium of the mass A B C will be impossible, if at any point, as $a\,m$, of the interior surface $a\,b\,c$, any other pressure exist besides that represented by p, or any other forces be in action besides those expressed by the coefficients of the variations in equation (2). For, at the upper surface, there are no forces in action but the principal forces, and the equilibrium will be impossible if other forces prevail in the interior parts besides the principal forces. On the other hand, the matter contained in the stratum between the two surfaces will attract every particle, as $a\,m$, situated in the interior surface. The attraction of the stratum is an indelible force not

to be destroyed, which will produce its full effect according to the figure and quantity of the attracting matter and the situation of the attracted point. The equilibrium will therefore be absolutely impossible, unless such a figure can be induced on the mass of fluid as will set free every particle in the surface $a\,b\,c$ from the attraction of the stratum. If such a figure can be found, every molecule of the mass will be urged by the principal forces only; because a surface such as $a\,b\,c$, at every point of which these forces alone will be in action, may be described through any interior molecule $a\,m$ arbitrarily assumed. We must therefore turn our attention to investigate such figures, if there be any, as will make the irregular attraction in the interior parts disappear, so as to leave the principal forces alone in action; for, unless this can be effected, the fluid cannot maintain a permanent form.

According to the notation we have used, if r denote the distance of $a\,m$ from G, $\mathrm{V}(r)$ will represent the sum of the quotients of all the molecules of the whole mass divided by their respective distances from $a\,m$; let $\mathrm{V}'(r)$ denote the same thing, relatively to the interior mass $a\,b\,c$, that $\mathrm{V}(r)$ does, relatively to the whole mass A B C; then $\mathrm{V}(r) - \mathrm{V}'(r)$ will denote the sum of the quotients of all the molecules of the stratum divided by their respective distances from $a\,m$. Take a point $(x + dx,\ y + dy,\ z + dz)$ in the surface $a\,b\,c$ infinitely near $a\,m$; and, differentiating in the surface, the expressions,

$$\frac{d.\big(\mathrm{V}(r) - \mathrm{V}'(r)\big)}{dx}, \qquad \frac{d.\big(\mathrm{V}(r) - \mathrm{V}'(r)\big)}{dy}, \qquad \frac{d.\big(\mathrm{V}(r) - \mathrm{V}'(r)\big)}{dz},$$

will be equal to the attractive forces of the stratum upon the particles of $a\,m$, in the respective directions of x, y, z: but, as we have shown, the equilibrium indispensably requires that these attractions be evanescent, so that we have these equations,

$$\frac{d.\big(\mathrm{V}(r) - \mathrm{V}'(r)\big)}{dx} = 0, \quad \frac{d.\big(\mathrm{V}(r) - \mathrm{V}'(r)\big)}{dy} = 0, \quad \frac{d.\big(\mathrm{V}(r) - \mathrm{V}'(r)\big)}{dz} = 0,$$

* The perpendicularity to the surface $a\,b\,c$, of the attraction of the stratum upon $a\,m$, is expressed by this equation,

$$\frac{d.\big(\mathrm{V}(r) - \mathrm{V}'(r)\big)}{dx}\,dx + \frac{d.\big(\mathrm{V}(r) - \mathrm{V}'(r)\big)}{dy}\,dy + \frac{d.\big(\mathrm{V}(r) - \mathrm{V}'(r)\big)}{dz}\,dz = 0;$$

and it is a consequence of the differential equations in the text. The neglect of this consideration, and the assumption that the level surfaces depend solely upon the outer surface in every case, is the great blemish of Clairaut's theory.

which are no other than the partial differentials of the equation,

$$V(r) - V'(r) = \text{constant.} \qquad (4)$$

This equation must hold at every point of every interior surface, such as $a\,b\,c$; and, as its differentials are separately equal to zero, it must not contain the coordinates of the surface. If such a figure can be induced on the mass of fluid as will possess the property expressed by equation (4), every particle of the mass will be urged by the principal forces alone, the equilibrium will be possible, and it will be determined in the very same manner as in the first problem.

We have now obtained a mathematical property that distinguishes the figures with which the equilibrium is possible from all others. We have also, in another place*, investigated the figures that alone possess this property; and it appears from what is there shown, that A B C can be no other but an ellipsoid, and that every interior surface, as $a\,b\,c$, is similar to the outer surface, and similarly posited about G.

Having demonstrated that the fluid in equilibrium must be an ellipsoid, it readily follows that the axis of rotation must be one of the three axes of the geometrical figure. For, as the axis of rotation passes through G, the centre of gravity, it is a diameter of the ellipsoid; and the centrifugal force being evanescent at the extremities of this diameter in the surface of the fluid, the only force in action at those points is the attraction of the mass of matter. But the whole force urging every particle in the outer surface of the mass in equilibrium, is perpendicular to that surface; wherefore, the attractive force of the ellipsoid is perpendicular to its surface at the extremities of the diameter about which the fluid revolves; and as there are no points on the surface of that geometrical figure at which the attraction of its mass is perpendicular to its surface, except the extremities of its three axes, it follows that with one or other of these, the axis of rotation of the fluid in equilibrium must coincide.

Let us now determine the relations between the axes of the ellipsoid and the centrifugal force. Of the three planes of the coordinates, one, which is perpendicular to the axis of rotation, is a principal section of the ellipsoid; and we may suppose that the other two coincide with the two remaining principal sections. We may therefore compute V (R) for a point in the surface; and by substituting this value in the equation,

* Phil. Trans. for 1824.

$$0 = \mathrm{V}(\mathrm{R}) + \frac{f}{2}(y^2 + z^2) - \mathrm{C},$$

and making the result coincide with the geometrical equation of the figure, we shall obtain the expressions of the axes in terms of the centrifugal force. But it will be more simple to use the differential equation,

$$-\frac{d.\mathrm{V}(\mathrm{R})}{dx}dx - \left(\frac{d.\mathrm{V}(\mathrm{R})}{dy} + fy\right)dy - \left(\frac{d.\mathrm{V}(\mathrm{R})}{dz} + fz\right)dz = 0,$$

which expresses the perpendicularity of the forces to the outer surface. The quantities,

$$-\frac{d.\mathrm{V}(\mathrm{R})}{dx}, \quad -\frac{d.\mathrm{V}(\mathrm{R})}{dy}, \quad -\frac{d.\mathrm{V}(\mathrm{R})}{dz},$$

are the attractive forces of the ellipsoid, urging a particle of the surface in directions parallel to the axes; and these forces, by the nature of the ellipsoid, are proportional to the coordinates of the point on which they act, and may be represented by $\mathrm{A}'x$, $\mathrm{B}'y$, $\mathrm{C}'z$, the coefficients A', B', C' being known quantities depending upon the ratios of the axes of the ellipsoid; wherefore, these values being substituted in the differential equation, we shall have,

$$\mathrm{A}'x\,dx + (\mathrm{B}' - f)y\,dy + (\mathrm{C}' - f)z\,dz = 0;$$

and by integrating,

$$x^2 + \frac{\mathrm{B}' - f}{\mathrm{A}'}y^2 + \frac{\mathrm{C}' - f}{\mathrm{A}'}z^2 = \text{constant}.$$

Now, if h, h', h'' represent the axes of the ellipsoid, h being that about which the fluid revolves, the equation of the surface of the figure will be,

$$x^2 + \frac{h^2}{h'^2}y^2 + \frac{h^2}{h''^2}z^2 = h^2;$$

and with this equation the foregoing one must be made to coincide. On account of the arbitrary constant, we have only to equate the coefficients of y^2 and z^2, and the resulting formulas may be thus written,

$$f = \mathrm{B}' - \frac{h^2}{h'^2}\mathrm{A}', \qquad f = \mathrm{C}' - \frac{h^2}{h''^2}\mathrm{A}'.$$

But, on examining the functions that A', B', C' stand for, it will readily appear that the expressions on the right side of the two formulas will not be positive,

and consequently they cannot be equal to f, unless $\frac{h^2}{h'^2}$ and $\frac{h^2}{h''^2}$ be both less than unit: and supposing that h is the least of the three axes, the two values of f will not be equal, unless $B' = C'$, and $h' = h''$, in which case both the formulas coincide in one, viz.

$$f = B' - \frac{h^2}{h'^2} A'.$$

In conclusion, it follows that the figure of the fluid in equilibrium is an oblate elliptical spheroid of revolution, of which the equation is

$$x^2 + \frac{h^2}{h'^2}(y^2 + z^2) h^2,$$

the mass turning about h the less axis, and the relation between the centrifugal force and $\frac{h}{h'}$ the ratio of the axes, being determined by the equation

$$f = B' - \frac{h^2}{h'^2} A'.$$

The complete solution of the problem is now brought to the discussion of this last equation; and as this is a question purely mathematical, but slightly connected with the physical conditions of the equilibrium, which we have undertaken to investigate, we shall refer to the Mécanique Céleste of Laplace and to the Théorie Analytique du Système du Monde of M. de Pontecoulant, in which works this point is amply treated.

The foregoing solution, being perfectly general, proves that the equilibrium is possible only when the elliptical spheroid is oblate at the poles. When the spheroid is oblong, and the axis of rotation h greater than the other axis h', the expression that must be equal to the centrifugal force is negative; and as that force is essentially positive, the equilibrium becomes impossible.

It will not be necessary to retrace the steps of the foregoing analytical process of reasoning, in order to show synthetically that the equilibrium will be secured if the conditions deduced be fulfilled. For, as soon as such a figure is found as will make the forces that actually urge every particle of the mass the same functions of the coordinates of their point of action, this problem comes under the hypothesis of the first one, and may be demonstrated in the very same manner.

The method of solution we have here followed may be applied to all problems concerning the equilibrium of a mass of fluid, when it is possible to form the equation of the outer surface; that is, when the forces in action at all the points of the outer surface are the same functions of the coordinates of those points, whatever geometrical figure the mass may be supposed to assume. This in reality comprehends every question that has hitherto occurred; and, as the conditions which we have laid down are necessary and sufficient for the equilibrium in every hypothesis of the forces that can be imagined, we shall not enter into any further discussion of this point.

9. The preceding analysis, by which we have investigated the figure of equilibrium of a homogeneous planet is direct and unexceptionable in point of rigour. It seems hardly possible to express simply in algebraic language, all the forces that urge the interior particles of the fluid; and this makes it necessary to have recourse to peculiar modes of reasoning for determining the figure of equilibrium. The problem, being one of great importance and difficulty, which has much engaged the attention of geometers, and which requires for its solution principles different from those that have so long passed current without suspicion of their accuracy, it may not be improper to add another investigation of it by a process of reasoning very different from the foregoing.

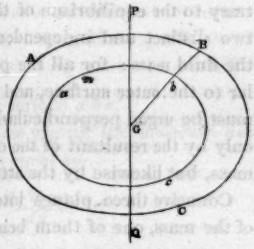
Second investigation.

We shall begin with laying down the following lemma. If a mass of homogeneous fluid, consisting of particles which attract one another inversely as the square of the distance, be in equilibrium when it revolves with a certain angular velocity about an axis; any other mass of the same fluid, the particles attracting by the same law, will be in equilibrium, if it have a similar figure, and revolve with the same rotatory motion about an axis similarly placed.

Take any two particles similarly placed in the two bodies, and having the same proportion to one another as the whole masses; it is proved in the Principia of NEWTON, and in the works of other authors, that the resultants of the attractive forces acting upon the particles, have similar directions, and are proportional to the linear dimensions of the two bodies. Further, the centrifugal forces urging the two bodies to recede from the axes of rotation, are proportional to the respective distances from the axes, that is, to the linear dimen-

sions of the two bodies. Wherefore the joint action of all the forces is to urge the two particles in similar directions with intensities proportional to the linear dimensions of the bodies. And as the same thing is true of all particles similarly situated in the two bodies, if there be an equilibrium in one case, there will be an equilibrium in the other; for the forces which urge the particles of one body are in no respect different from the forces which urge the particles of the other, except in being all increased or all diminished in the same given proportion.

This lemma being premised, let A B C represent a mass of homogeneous fluid in equilibrium, by the attraction of its particles in the inverse proportion of the square of the distance, and a centrifugal force caused by revolving about the axis P Q. The axis P Q will pass through G, the centre of gravity of the mass. For, abstracting from any motion or force common to all the particles, that centre may be considered at rest; and, as the attractive forces of the particles balance one another at that point, the centrifugal force must likewise vanish at the same point.



Let any radius G B, drawn from the centre of gravity to the surface of the fluid, be divided in a given proportion at b; and supposing it to turn round G so as to be directed successively to all the points in the outer surface of the fluid, the radius G b, being always the same part of G B, will describe an interior surface similar to the outer one, and similarly posited about G. And because the whole mass A B C is in equilibrium, it follows from the lemma that the interior mass $a\,b\,c$, which is similar to the whole mass, and revolves with it about the common axis P Q, will be separately in equilibrium, supposing the exterior stratum of matter were taken away or annihilated.

In the interior surface $a\,b\,c$ assume any molecule $a\,m$: the forces that act upon $a\,m$ are; first, the resultant of the centrifugal force and the attraction of the mass $a\,b\,c$; secondly, the attraction of the stratum of fluid between the two surfaces. Because the interior body of fluid $a\,b\,c$ is separately in equili-

brium, the first of these forces, namely, the resultant of the centrifugal force and the attraction of the mass $a\,b\,c$, is perpendicular to the surface $a\,b\,c$, and destroyed by the resistance of the fluid within that surface; and from this it follows that the attraction of the stratum upon $a\,m$, must likewise be perpendicular to the same surface. For, if it acted obliquely to the surface $a\,b\,c$, it might be resolved into two partial forces, one perpendicular, and the other parallel, to the plane touching the surface; and as there is no obstacle to oppose the latter force, it would cause the molecule $a\,m$ to move, which is contrary to the equilibrium of the whole mass A B C. It appears therefore that two distinct and independent conditions are required for the equilibrium of the fluid mass: for all the particles situated in any interior surface $a\,b\,c$ similar to the outer surface, and similarly posited about the centre of gravity G, must be urged perpendicularly to the surface in which they are contained, not only by the resultant of the centrifugal force and the attraction of the interior mass, but likewise by the attraction of the exterior stratum of fluid.

Conceive three planes intersecting at right angles in the centre of gravity of the mass, one of them being perpendicular to the axis of rotation P Q: let x, y, z represent the coordinates of the molecule $a\,m$, and $r = \sqrt{x^2 + y^2 + z^2}$, its distance from G, x being parallel to P Q; and put $\mathrm{V}(r)$ for the sum of the quotients of all the molecules of the whole mass A B C, divided by their respective distances from $a\,m$: further, let $\mathrm{V}'(r)$ denote the same thing relatively to the interior mass $a\,b\,c$, that $\mathrm{V}(r)$ does relatively to the whole mass A B C: then $\mathrm{V}(r) - \mathrm{V}'(r)$ will be the sum of the quotients of all the molecules of fluid contained in the stratum between the two surfaces, divided by the respective distances of the molecules from $a\,m$. According to the known properties of this function, the partial attractions of the stratum upon $a\,m$, in the directions of x, y, z, and tending to lengthen these lines, will be respectively equal to

$$\frac{d.\left(\mathrm{V}(r) - \mathrm{V}'(r)\right)}{dx}, \quad \frac{d.\left(\mathrm{V}(r) - \mathrm{V}'(r)\right)}{dy}, \quad \frac{d.\left(\mathrm{V}(r) - \mathrm{V}'(r)\right)}{dz}.$$

Take any point $(x + dx,\ y + dy,\ z + dz)$ in the surface $a\,b\,c$, at the infinitely small distance ds from $a\,m$: then the resultant of the foregoing attracting forces in the direction of ds will be equal to

$$\frac{d.\left(\mathrm{V}(r)-\mathrm{V}'(r)\right)}{dx}\cdot\frac{dx}{ds}+\frac{d.\left(\mathrm{V}(r)-\mathrm{V}'(r)\right)}{dy}\cdot\frac{dy}{ds}+\frac{d.\left(\mathrm{V}(r)-\mathrm{V}'(r)\right)}{dz}\cdot\frac{dz}{ds};$$

and this resultant must be equal to zero in whatever direction ds is drawn, if the attraction of the stratum upon am be perpendicular to the surface abc. Wherefore we have,

$$\frac{d.\left(\mathrm{V}(r)-\mathrm{V}'(r)\right)}{dx}dx+\frac{d.\left(\mathrm{V}(r)-\mathrm{V}'(r)\right)}{dy}dy+\frac{d.\left(\mathrm{V}(r)-\mathrm{V}'(r)\right)}{dz}dz=0: \quad (5)$$

and, by integrating,

$$\mathrm{V}(r)-\mathrm{V}'(r)=\text{Constant}, \quad (6)$$

which equation must be true at every point in the surface abc.

Again, the attractive forces of the interior mass urging the molecule am inwards in the direction of x, y, z, are respectively equal to

$$-\frac{d.\mathrm{V}'(r)}{dx}, \quad -\frac{d.\mathrm{V}'(r)}{dy}, \quad -\frac{d.\mathrm{V}'(r)}{dz}.$$

Let f denote the centrifugal force at the distance unit from the axis of rotation; and, the distance of am from the same axis being $\sqrt{y^2+z^2}$, the centrifugal force of the particles of am will be $f\sqrt{y^2+z^2}$; and the resolved parts of this force acting in the prolongations of y and z, will be fy and fz. Wherefore the total accelerating forces urging am in the directions of x, y, z, and tending to shorten these lines, are respectively,

$$-\frac{d.\mathrm{V}'(r)}{dx}, \quad -\left(\frac{d.\mathrm{V}'(r)}{dy}+fy\right), \quad \left(\frac{d.\mathrm{V}(r)}{dz}+fz\right):$$

and, the condition that the resultant of these forces is perpendicular to the surface abc, is expressed by this differential equation,

$$-\frac{d.\mathrm{V}'(r)}{dx}dx-\left(\frac{d.\mathrm{V}'(r)}{dy}+fy\right)dy-\left(\frac{d.\mathrm{V}'(r)}{dz}+fz\right)dz=0 \quad (7)$$

In the equations (5) and (7) the forces expressed by the co-efficients of the differentials, act on the same particles and have opposite directions in the same lines; wherefore by subtracting the former from the latter, we have,

$$-\frac{d.\mathrm{V}(r)}{dx}\,dx-\left(\frac{d.\mathrm{V}(r)}{dy}+fy\right)dy-\left(\frac{d.\mathrm{V}(r)}{dz}+fz\right)dz=0, \qquad (8)$$

in which the co-efficients of the differentials express the whole forces urging the molecule in the directions of x, y, z.

It is obvious that the equations (7) and (8) must be identical; for they are both true at every point of the same surface $a\,b\,c$. But if the co-efficients of the differentials of these two equations be identical, the like co-efficients in the equation (5) must be separately equal to zero; and this proves that the co-ordinates of the surface $a\,b\,c$ do not enter into the equation (6), which therefore contains such quantities only as remain invariably the same at all the points of that surface.

The equations (7) and (8) being identical, the latter will belong indifferently to all the similar surfaces in the interior parts, and to the outer surface which is their limit. Wherefore, if for the sake of distinction we suppose that r becomes R at the upper surface, we shall obtain the equation of that surface by integrating, viz.

$$\mathrm{C}=\mathrm{V}(\mathrm{R})+\frac{f}{2}(y^2+z^2). \qquad (9)$$

The integral of (8) will likewise give the equation of any of the interior surfaces, as $a\,b\,c$, viz.

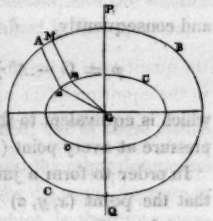
$$p=\mathrm{V}(r)+\frac{f}{2}(y^2+z^2)-\mathrm{C}, \qquad (10)$$

the quantity C being absolutely constant in all circumstances, and the same as in the equation of the upper surface, and p being a new quantity which is constant when the co-ordinates are taken in the surface $a\,b\,c$, but varies when the co-ordinates belong to any point of the mass not contained in that surface. At the upper surface p vanishes; it changes its value in passing from one of the interior surfaces to another; and it is evidently the hydrostatic pressure at every point of the mass, because $-\frac{dp}{dx}$, $-\frac{dp}{dy}$, $-\frac{dp}{dz}$, are equal to the co-efficients of the differentials in equation (8) and to the accelerating forces which oppose and destroy the variation of pressure. The equations (6) and (9) and (10) at which we have arrived by this new train of reasoning are the very same with the equations (4) and (1) and (3) of the first investigation; and as the

remainder of the solution is deduced entirely from these equations, it would be superfluous to repeat here what has already been fully explained. The same procedure as in the first investigation will prove, that the figure of the fluid in equilibrium is exclusively an oblate elliptical spheroid of revolution turning about the less axis h, and that the ratio $\frac{h}{k}$ of the two axes is derived from the centrifugal force by means of the equation

$$f = \mathrm{B}' - \frac{h^2}{k^2}\,\mathrm{A}'.$$

10. The level surfaces of the mass in equilibrium are properly the interior surfaces similar to the outer surface, and similarly posited about the common centre. Such surfaces agree with CLAIRAUT'S definition; for they are perpendicular to the resultant of the forces urging the particles contained in them, as appears from the differential equation (8), which is common to them all. But as every particle within the mass is acted upon by several forces, it may become a question whether there are not other interior surfaces besides those similar to the outer one, which possess the properties of being equably pressed, and of being perpendicular to the resultant of the forces in action. It is this point that we are now to investigate.



Suppose that A B C represents an oblate elliptical spheroid of homogeneous fluid in equilibrium by revolving about the axis P Q; and let $a\,b\,c$ be an oblate elliptical spheroid within A B C, the centres, the less axes, and the equators of the two figures coinciding: taking any particle (x, y, z) of the interior mass $a\,b\,c$, the attractions of the whole mass A B C urging the particle in the respective directions of the co-ordinates, may, as before, be represented by

$$\mathrm{A}'x, \qquad \mathrm{B}'y, \qquad \mathrm{B}'z:$$

and in like manner the attractions of the interior mass $a\,b\,c$ upon the particle, may be denoted by

$$\mathrm{A}''x, \qquad \mathrm{B}''y, \qquad \mathrm{B}''z:$$

and the attractions of the matter between the two surfaces upon the particle, will be

$$(A' - A'')\,x, \qquad (B' - B'')\,y, \qquad B' - B'')\,z.$$

As these forces act upon every particle of the mass $a\,b\,c$, they will cause an internal pressure; let p' denote the hydrostatic pressure at the point (x, y, z) caused by the attraction of the external matter; then, by the general theory, we shall have

$$dp' + (A' - A'')\,x\,dx + (B' - B'')\,(y\,dy + z\,dz) = 0;$$

and, by integrating,

$$p' = C' - (A' - A'')\,\frac{x^2}{2} - (B' - B'')\,.\,\frac{y^2 + z^2}{2}. \qquad (11)$$

Further, the joint effect of the centrifugal force and the attraction of the whole mass A B C upon the particle (x, y, z) in the respective directions of the coordinates, is expressed by these forces,

$$A'x, \qquad (B' - f)\,y, \qquad (B' - f)\,z;$$

and if p be the pressure thence arising, we shall have

$$dp + A'\,x\,dx + (B' - f)\,(y\,dy + z\,dz) = 0;$$

and consequently,

$$p = C - A'\,\frac{x^2}{2} - (B' - f)\,.\,\frac{y^2 + z^2}{2} \qquad (12)$$

which is equivalent to the equation (10), and expresses the whole hydrostatic pressure at every point (x, y, z) within the mass A B C.

In order to form a just notion of the pressures p and p', we shall suppose that the point (x, y, z) is in the interior surface, at $a\,m$: conduct a narrow canal from G to $a\,m$, and continue it outward to the upper surface of the fluid, at A M. Now p is the effort of all the molecules in the canal A $a\,m$ M produced by all the forces that urge them along the canal; and p' is the effort of the canal a G m caused by the attraction of the matter between the two surfaces upon the particles contained in the canal. The pressure p is always directed inward; but the direction in which p' acts will depend upon the na-

ture of the interior spheroid. If it be more oblate than the exterior spheroid A B C, A'' will be greater than A', and the attraction $(A' - A'')\,x$ tending from the equator, the pressure of the canal $a\,G\,m$ will be outward and opposed to that of the canal $A\,a\,m\,M$. On this supposition, therefore, the whole action of the matter exterior to the spheroid $a\,b\,c$ will cause a pressure upon the molecule $a\,m$, equal to $p - p'$. By subtracting the equations (11) and (12) we get

$$p - p' = C - C' - A''\frac{x^2}{2} - (B'' - f)\cdot\frac{y^2 + z^2}{2}: \qquad (13)$$

and we have now to inquire whether a spheroid can be found that will satisfy this equation, on the supposition that $p - p'$ is the same at all the points of the surface of the spheroid.

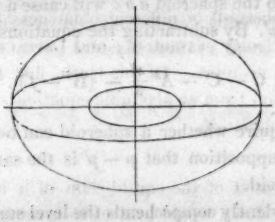
The equation (13) evidently comprehends the level surfaces, which are similar and similarly situated to the upper surface A B C: for, on the supposition that the figures are similar, we have $A' = A''$, $B' = B''$, $p' = C'$, and the equation (13) is identical to the equation (12) which, by giving different values to p, determines all the level surfaces. The equation (13) is similar in its form to the equation (12), A'' and B'' being the same functions of the excentricity of the spheroid $a\,b\,c$, that A' and B' are, of the excentricity of the spheroid A B C; and the centrifugal force f enters alike into both equations. It is therefore evident that the solution of the latter, supposing p constant, and the solution of the former supposing $p - p'$ constant are both contained in the equation,

$$f = B' - \frac{h^2}{k^2}A':$$

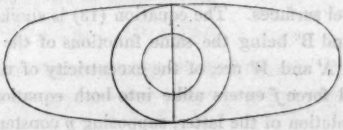
and, as from this two values of $\frac{h^2}{k^2}$ are in general obtained, one of these results determines the spheroid A B C and its level surfaces, and the other determines the interior spheroid $a\,b\,c$, the surface of which sustains the same pressure at every point by the action of the exterior fluid, and which is therefore separately in equilibrium.

There is this difference between the level surfaces and the other surfaces of equable pressure, that the former spread through the whole mass and ultimately coincide with the upper surface, whereas the latter, on account of the dissimilarity of figure, are confined to a part of the mass. Of the two spheroids

answering to the same centrifugal force, when the exterior one is the less oblate, the greatest interior surface of equable pressure, which is not a level surface, stands upon the equator; and the rest are within this, similar and concentric to it, as in this figure



When the exterior spheroid is the more oblate of the two, the greatest interior surface is described on the less axis, and the rest are similar and concentric to it, as thus,



When the centrifugal force f has a certain relation to the attractive force, the two dissimilar spheroids A B C and $a\ b\ c$ coincide in one; and in this case there are no interior surfaces of equable pressure except the level surfaces.

It has now been demonstrated that, in every oblate spheroid in equilibrium by a rotatory motion, there are two sets of interior surfaces equably pressed by the action of the exterior fluid; and, in consequence, that there are two different figures of equilibrium, and only two answering to the same velocity of rotation. But in the hypothesis of the first problem of this paper, and according to the theory of CLAIRAUT, which as far as regards a fluid entirely at liberty, is equivalent to that problem, there is in every case of equilibrium, only one set of interior surfaces equably pressed by the exterior fluid; and this is an incontrovertible proof that the theory of the French geometer is insufficient for determining the figure of equilibrium of a homogeneous planet in a fluid state.

Maclaurin first demonstrated synthetically the equilibrium of an oblate elliptical spheroid when it revolves about the less axis with a certain angular velocity. In examining the equation of the surface of the fluid, D'Alembert discovered that it admitted of being solved more than one way, that is, he found that there are spheroids of different oblateness which will be in equilibrium with the same velocity of rotation; and Laplace proved that there are two such spheroids and no more. Of this truth, first made known merely as a mathematical deduction from an algebraic equation, we have here attempted to give the physical explanation.

Having now fully treated of the equilibrium of a homogeneous fluid, the order of discussion laid down would lead us to investigate that of one of variable density; but the length of this paper makes it advisable to reserve this part of our subject for another occasion.

VIII. *On a simple Electro-chemical Method of Ascertaining the Presence of different Metals; applied to detect minute quantities of Metallic Poisons.* By Edmund Davy, *F.R.S. M.R.I.A. &c. Professor to the Royal Dublin Society.*

Read November 25, 1830.

1. *Introduction.*

It is now nearly a quarter of a century since the late Sir Humphry Davy, by a train of masterly researches, developed the general principles of electro-chemical action, which subsequently led him to many fine discoveries and important practical applications. Some years since, I repeated most of the interesting experiments noticed in his excellent Bakerian Lecture "On the chemical agencies of Electricity*." On the decomposition of metallic salts by the Voltaic battery, Sir Humphry is very brief. He clearly ascertained, however, that "when metallic solutions were placed in the circuit, metallic crystals or depositions were formed on the negative surface †;" and that "the metals passed towards the negative surface, like the alkalies, and collected round it †." In the course of my experiments on this subject, phenomena occurred which led me to think that some novel results might be obtained by instituting a series of experiments on metallic salts, using as a Voltaic arrangement the feeble power produced by the contact of small slips of different metals, with solutions of the common metallic salts. Operating in this manner, I could readily detect very minute quantities of different metals, coat platina with gold, silver, copper, &c., or cover gold with a surface of these metals, and tin, copper, brass, iron, &c. Several of those facts I have been in the habit of bringing forward and illustrating in my annual courses of lectures delivered both in the Royal Cork Institution, and in the Royal Dublin Society. Circumstances which it is unnecessary to mention, have hitherto prevented me from

* Philosophical Transactions of the Royal Society, 1807. † Ibid.

giving greater publicity to those facts. In the course of the present summer my attention has been directed to apply similar means to the detection of metallic poisons, a subject of acknowledged and increasing importance; and the results I have obtained, and now beg leave to submit to the Society, appear to me both novel and interesting, and afford, if I mistake not, means more simple, delicate, and effectual, than any at present known for detecting the common metallic poisons.

The fear of trespassing too much on the time of the Society, induces me to limit the present paper to one part only of the subject. At no distant period I promise myself the pleasure of communicating the remaining part, which will embrace the different electro-chemical experiments I have made on the other metals and their compounds, together with the application of the facts to the processes of gilding, silvering, tinning, &c.

In the following pages I shall notice the simple electro-chemical apparatus, (or electro-chemical method as I shall call it,) employed in my experiments; offer proofs of its efficacy to detect different metals, particularly metallic poisons; adduce instances of the extreme delicacy and facility of the method; and lastly, show, by similar evidences, that its accuracy is not impaired by the presence of organic substances whether vegetable or animal, or mixtures of both; and that the method is therefore applicable to the detection of metallic poisons in all cases. I shall be under the necessity of making some minute (and I fear tedious) details, which I trust will be excused, as they are closely connected with the elucidation of the subject.

It forms no part of my object to examine the numerous known methods of detecting metallic poisons. Experience, I may presume, has made me tolerably familiar with the details of them. The electro-chemical method here proposed appears to me to rival the very best of them in point of accuracy, whilst in facility, simplicity and delicacy, it seems superior to them all.

2. *Of the Electro-chemical Apparatus.*

The electro-chemical apparatus I used was of the simplest kind. It consisted of two different metals, generally zinc and platina, which, according to Sir H. Davy, form the most efficient combination; the zinc being positive, and the platina negative, with regard to all the other metals. The zinc was usually

in the state of foil, or of thin sheet. The platina was in some cases a small crucible, or a spatula having a spoon at the end of it, but more frequently platina foil was employed. The foil was about two inches long, and two thirds of an inch wide. The zinc foil varied from about one third to one eighth of the size of the platina foil. The size and thickness of either metal may vary indefinitely, without materially altering the results. This simple arrangement is most easily applied to the decomposition of a great number of metallic compounds. It is only necessary to mix a drop or two of acid with a little of such compounds, whether solid or fluid, and apply the zinc foil, when the platina will be soon coated with the reduced metal. Solutions of many metallic salts do not require the addition of acid. I may remark that the small slips of platina and zinc foil are very convenient for many experiments on metallic poisons; as where the object is to ascertain the presence of arsenic or mercury, in a fluid in which it may exist in considerable quantity; or to determine whether any powder contain either of those metals in combination. One slip of platina will answer for an indefinite number of such experiments; one slip of zinc, too, may be employed for many experiments. It is only necessary either to dip the end, after being used, into a little water, and wipe it, or to cut the mere point off at once. The platina spatula with a spoon at the end of it, is well adapted for concentrating, or boiling nearly to dryness, fluids which may contain metallic poisons, but in such minute quantity, as to render concentration indispensably necessary to the success of the electro-chemical method of detecting them. The small platina crucible is a necessary appendage to the apparatus, in cases, where from previous trials by the platina foil, or spoon, and zinc, the existence of a metallic poison has been proved in a fluid or solid, in order to collect it in sufficient quantity, and exhibit it in a separate state.

3. *Experimental proofs of the efficacy of the Electro-chemical method to detect different metals, and especially metallic poisons.*

Solutions of gold, silver, mercury, copper, tin, lead, &c. are not decomposed, as is well known, by platina; but if a drop of each of those metallic salts, containing excess of acid, be severally placed on a surface of platina, and a slip of zinc brought in contact with both, each salt will be decomposed, and the

respective metal deposited on the platina. These are a few instances of the apparent reduction of metallic salts on platina by the agency of Voltaic electricity. It would be extremely easy to extend the list to an indefinite number of other cases, did I not fear to anticipate details which more properly belong to the subsequent part of the subject.

The compounds of arsenic, mercury, lead and copper afford the principal metallic poisons; and a knowledge of their properties, and of the best means of detecting them, derives interest from many considerations, but is particularly valuable from the paramount importance justly attached to chemical evidence in cases of accidental or intentional poisoning. I shall proceed to offer proofs of the efficacy of the electro-chemical method to detect the compounds of those metals in the order in which they have been enumerated.

Compounds of Arsenic.—{ Greyish black arsenic, protoxide of Berzelius, or fly-powder. / White oxide of arsenic, or arsenious acid.

When half a grain, or less, of either of those solid compounds is placed on a slip of platina foil, mixed with a drop or two of muriatic acid*, and the zinc applied†, the arsenic will presently be reduced to the metallic state; one part will be deposited on the platina, and the other part mixed in thin filaments with the fluid. The surface of the platina will become iridescent or exhibit variegated colours (resembling heated steel), as blue with tints of red, yellow‡, &c. Much of the arsenic is thus strongly attached to the platina, and cannot be removed by wiping or rubbing it with the finger or a cloth, nor by cold strong muriatic or sulphuric acid, nor by hot solutions of caustic alkalies; but it instantly disappears on being touched with the smallest drop of

* By "muriatic acid" in this paper, the common strong acid of commerce is meant. I have used it in preference to the pure acid, from the facility with which it can be procured.

† By the words "zinc applied," used here and in other places, is meant bringing the zinc in contact both with the platina and the substances to be acted on; or moving the zinc about on the platina, which seems the readiest way of effecting the reduction of many metallic compounds.

‡ Diluted muriatic acid occasions analogous effects, but they are more slowly produced, probably because the undiluted acid is a better solvent of the arsenical compounds, and a better conductor of electricity. Like effects occur with strong sulphuric acid, if the zinc be moved about on the platina, but they are not produced with strong nitric acid or aquafortis, and only very gradually by these acids when diluted.

strong nitric acid, or of the aquafortis of commerce. The arsenic is also readily removed by exposing the foil to the moderate heat of a spirit lamp, when it rises in the form of arsenious acid; but previous to this effect, the alliaceous, or garlick-like odour, so characteristic of this metal, is strongly produced *, and the surface of the platina remains unaltered.

The following arsenical compounds, when exposed to similar treatment, afford analogous results, viz.

Arsenious acid in solution.

Arsenites of potash, lime, &c.

Arsenic acid.

Arseniates of potash, lime, &c.

Chloride of arsenic.

Sulphurets of arsenic, obtained by passing sulphuretted hydrogen gas through solutions of arsenious and arsenic acids.

In cases where the quantity of any arsenical compound in solution is very minute, the fluid should be concentrated, or boiled nearly to dryness, previous to the addition of the muriatic acid, and subsequent trial on the platina foil by zinc.

Some arsenical compounds require a treatment somewhat different from those already enumerated, in order to exhibit the arsenic in a satisfactory manner. This is the case with the native sulphurets of arsenic, which being scarcely acted on by muriatic acid, require previous treatment with nitric acid. Thus, a little realgar or orpiment, in powder, was mixed with a drop of strong nitric acid in the platina spoon, heated, and the excess of acid expelled; a drop or two of muriatic acid being incorporated with the residual substance, the zinc was applied, and the arsenic readily reduced on the spoon, exhibiting its characteristic appearance. Arsenical pyrites, too, requires the previous addition of nitric acid. In operating on such arsenical compounds as require the use of nitric acid, as little as possible should be employed, and any excess ex-

* Dr. Christison, in his valuable work "On Poisons," proposes to discard this test altogether, chiefly, it would seem, from its being obscured entirely by the presence of a very small portion of vegetable or animal matter; but as this objection does not apply to the electro-chemical mode of detecting arsenic, the alliaceous odour is regarded, especially in these experiments, as a striking character, exhibited by no other metal, as far as my experience extends.

pelled, as a minute quantity of this acid retards the reduction of the arsenic, or re-dissolves it.

The metallic arsenites and arseniates, where two metals are present, appear also in general to require a modified treatment, according to the object in view, and seem to offer proofs of the elegance of this mode of detecting metals. To give an instance or two, in the case of the arsenite or arseniate of copper. If a little of either of these compounds be dissolved in a few drops of muriatic acid, by heat, in the platina spoon, and a few tolerably quick contacts be made with zinc foil, the arsenic only will be reduced; part of it will be deposited on the spoon, and part will remain in the fluid as a dark grey or blackish substance.

If a little of either of the above compounds be dissolved in a few drops of diluted nitric acid, (consisting of one volume strong acid to three of water,) in the platina spoon; boiled nearly to dryness, and water added just enough to obtain a solution; if the point of a slip of zinc be now applied to the centre of the bulb of the spoon, the copper alone will presently be reduced on the platina, exhibiting an unusual metallic lustre, and forming a circle round the zinc. If the zinc be now removed, the copper will be re-dissolved by the slight excess of acid present. If the contact of the zinc be continued, in a short time the arsenic will form a circle on the platina, round the zinc, whilst beyond it the copper will make its appearance.

I found by repeated experiments, that the arsenic precipitated by zinc on platina foil from solid arsenious acid, or from arsenical solutions, and muriatic acid, could be readily obtained in the form of arsenious acid, by coiling the dried foil, and exposing it to the heat of a spirit lamp, in a tube closed at one end, or open at both, and stopped with a cork or corks, so as to allow the expanded air to escape.

With a view to gain some approximation as to the actual quantity of arsenic that could be detected by the electro-chemical method, and to ascertain if it could be procured from the platina in the metallic state, I placed on a new slip of platina weighing 22.14 grains, five drops of an aqueous solution of arsenious acid, and three drops of muriatic acid; a slip of zinc being applied, the arsenic was soon reduced, and much of it adhered to the platina, which, after being washed in pure water and dried, acquired an in-

crease of $\frac{1}{300}$th part of a grain *. The platina was put into a small retort, which was then exhausted, twice filled with pure carbonic acid gas, and heated over a jar of the same gas, until all the arsenic rose and condensed in the upper part of the bulb of the retort as an extremely delicate whitish film, exhibiting no metallic lustre, even by the aid of a magnifying glass; nor could this be expected from the minute quantity of arsenic present, and the extent of surface over which it was spread by sublimation. The platina was found perfectly clean, and of precisely the same weight as at first. Some pure water was put into the retort, and occasionally agitated in contact with the sublimate, but after thirty hours a recent solution of sulphuretted hydrogen occasioned no change in the water. The actual quantity of arsenic attached to the platina in the foregoing experiment, was ascertained to be $\frac{1}{300}$th part of a grain. But this is very far from conveying a just idea of the degree to which this mode of detection may be carried: for a single drop of the aqueous arsenious acid would have afforded ample evidence of the arsenic as it respected colour, insolubility in muriatic and sulphuric acids, alliaceous odour, and volatility; which would give the $\frac{1}{3000}$th part of a grain. Even this very minute quantity gives us by no means the extreme limits to which this truly microscopic method of detecting metals may be carried. It is, however, quite unnecessary to pursue the subject further.

In cases when the small platina crucible was used, the results were equally delicate, and much more simple and satisfactory. I put a single drop of aqueous arsenious acid, with about an equal bulk of muriatic acid, into the crucible; the zinc being applied for an instant, the arsenic was reduced on the platina. The crucible was then rinsed with pure water, dried, covered with a piece of plate glass, and heated with a spirit lamp; arsenious acid rose and condensed on the glass, whilst the surface of the crucible remained unchanged. The results were precisely similar, when a single drop of arsenic acid, of arsenite and of arseniate of potash, were exposed to similar treatment.

I put $\frac{1}{100}$th part of a grain of solid arsenious acid into a very small platina crucible, which I coated with gold on the inside, and mixed with it about half

* In my experiments I used a very delicate balance of Robinson's construction, which turns with the $\frac{1}{1000}$th part of a grain, when loaded with one hundred grains.

a drop of muriatic acid. A small slip of zinc being applied, the spot became of a dark steel-grey colour with a tint of green at its edge. Some pure water being put into the crucible, some extremely minute films of metallic arsenic appeared on the surface of it. The crucible, after being washed and dried, was heated, when the arsenic rose, leaving the gold surface unaltered.

Into the crucible used in the preceding experiment, I put $\frac{1}{10}$th of a grain of solid arsenious acid, and dissolved it in about five drops of muriatic acid. I then applied a slip of zinc for about half a minute; as soon as the contact of the zinc was made, the variegated colours from the arsenic were beautifully produced at the bottom of the crucible, an effect which was succeeded by a violent action, and all the arsenic was reduced. The crucible was now filled with pure water, and numerous steel-grey coloured filaments of metallic arsenic floated on the surface; some being collected on a slip of platina and heated, the alliaceous odour was strongly produced, and they were dissipated in a white vapour. The crucible being washed, and a few drops of diluted muriatic acid put into it, it was rinsed in pure water and dried, when its gold surface was so completely coated with arsenic of a dark steel-grey colour, that no vestige of the gold at the bottom of the crucible could be seen, even with the aid of a magnifying glass, though the surface was full of little inequalities. The arsenic on the crucible did not sensibly tarnish by exposure to the air for some days; a portion of it was then expelled by a heat of nearly 400° Fahr., when the gold became partially visible; the remainder of the arsenic continued of a steel-grey colour.

Though it seems unnecessary to bring forward more experiments in proof of the efficacy of the electro-chemical method to detect arsenic, it may be proper to recur to the evidence that arsenical compounds are by this method reduced to the metallic state. The experiments with the platina gilt crucible afford the most unequivocal proofs of metallic arsenic; whilst those with the platina foil and spoon, though seemingly more ambiguous, will, on examination, I presume, be found scarcely less satisfactory. Thus the partial oxidation of the surface exhibited in variegated colours; the strong cohesion of the arsenic to the platina, are characteristics of the metal. The alliaceous odour (in cases where no deoxygenating substance can be presumed to exist) is admitted to belong only to the metal; insolubility in strong muriatic acid, is a property of no

known oxide of arsenic, but of the metal. To which may be added the facts, that I have repeatedly coated the arsenic, deposited on platina, with copper, and also with mercury; and removed both, without affecting the arsenic. All these circumstances seem to prove the accuracy of the statements made on the subject.

Though zinc and platina were the metals used in the electro-chemical experiments on arsenic already noticed, I have also employed several other metals which generally afforded analogous results. Other metals might of course be substituted for zinc and platina, but as far as my experiments have extended I give these a decided preference. Iron and tin are much slower in their operation than zinc. Brass and copper are readily coated with metallic arsenic; but the objections to the use of the common metals as substitutes for platina, arise from the facility with which they are acted on by heat, air, acids, &c. and the difficulty with which arsenic is separated from them without injuring their surfaces. The advantage of colour which gold has over platina, when a grey metal, as arsenic, is to be reduced in contact with it, is cheaply purchased by simply gilding part of the inside of the platina crucible.

Compounds of mercury.—The compounds of mercury in general are readily reduced by being placed on a slip of platina foil, mixed with a drop of diluted * aquafortis or muriatic acid, and the zinc applied, when the mercury is soon either partly attached loosely, or amalgamated with the platina, and partly combined with the zinc. This is the case with the black and red oxides, white precipitate, the acetate, subsulphate, cyanuret, fulminate, &c. The cyanuret, however, appears to be reduced more readily, and with greater lustre, when muriatic acid, undiluted, is employed, and the acetate and subsulphate best, when diluted aquafortis is used. The compounds of mercury soluble in water, in general require not the addition of acid.

From the importance of the compounds of mercury with chlorine, our details respecting them may be more minute.

Corrosive sublimate.—If a drop of an aqueous solution of corrosive sublimate be put on a bright surface of copper, it will soon, as is well known, render the copper of a greyish black colour. If a similar experiment be made and

* By "diluted" as applied to aquafortis and muriatic acid in this paper, is meant, the strong acids of commerce, to which an equal bulk of water has been added.

the zinc applied, the mercury will presently be reduced to the metallic state, and the copper will be whitened. The results will be analogous, if similar experiments are made with solutions of corrosive sublimate in alcohol and ether, provided a little water be employed when the zinc is applied, or if platina be substituted for the copper.

The following is a beautiful and extremely delicate mode of reducing corrosive sublimate, and of obtaining the mercury in a sensible form; and it may of course be applied to other compounds of mercury. Put a drop or two of an aqueous solution of corrosive sublimate into a small platina crucible, add about an equal bulk of muriatic acid, and apply the zinc for a short time, when the mercury will be reduced; part of it on the platina, which will appear of a brighter white colour, and part of it will amalgamate with the zinc, whitening it and making it brittle at the point of contact. Wash the crucible with pure water, dry, and cover it with a piece of plate glass, and heat it with a spirit lamp: the mercury will rise and be condensed as an extremely fine white powder without metallic lustre, and it may be easily collected from the glass by the finger, so as to be exhibited in minute globules, visible by a high magnifying power, even when a single drop only of the solution has been operated on.

I repeated the preceding experiment in the platina gilt crucible. The instant the zinc was applied the surface of the gold was whitened. Afterwards the mercury was collected by sublimation on glass.

The solution of corrosive sublimate may of course be readily reduced on platina when the zinc is applied without the addition of any acid, as in the ingenious mode proposed some time since by Mr. Sylvester, and afterwards simplified by Dr. Paris. I am disposed to prefer the use of platina to that of gold, not only on account of the great difference in their commercial value, but because I found the surface of gold tarnished after the mercury had been sublimed from it, which was not the case with the platina. I did not, indeed, ascertain if the gold I employed contained any alloy;—it was supposed to be pure.

A very simple mode of detecting corrosive sublimate, whether solid or dissolved in water, alcohol or ether, which I have not seen any where noticed, is, to put on a bright surface of copper, a bit of the solid compound, or a drop of

its solution, and to add a drop of muriatic acid to it, when the mercury will be immediately reduced, and the copper whitened. Similar effects also take place when a number of other compounds of mercury are similarly treated, as the oxides, cyanuret, the yellow precipitate from corrosive sublimate by the fixed alkalies, the white precipitate by ammonia, &c. Acetate of mercury does not yield to such treatment, unless heat be applied, when the copper is readily whitened; but with diluted aquafortis, the acetate is easily reduced on copper.

Calomel is slowly decomposed in a platina crucible, when mixed with diluted muriatic acid, and the zinc applied; the reduced mercury is deposited on the platina and combined with the zinc, or in case gold is present, it is whitened.

The following is a much readier and better mode of treating calomel. Mix a little of this compound, with a few drops of diluted aquafortis, and boil for an instant or two in a small platina crucible, add a little water, and apply the zinc; the mercury will be readily reduced. Break the contact of the zinc with the platina, and the bright mercurial surface of the latter will at once become tarnished. Restore the contact, and the lustre of the surface will reappear; and these changes will occur as often as the contact is broken and renewed. Add some pure water whilst the zinc is in contact, and the surface will remain bright. The mercury may be collected by sublimation, as in the case of corrosive sublimate. Operating in this manner, the mercury from a very small quantity of calomel may be obtained. Calomel may also be readily detected, by placing it on a clean surface of copper, mixing a little muriatic or diluted muriatic acid with it, and applying a gentle heat; when the mercury will be instantly reduced, and the copper whitened.

Compounds of lead.—The soluble compounds of lead, in general, like those of mercury, are readily reduced to the metallic state when placed on a surface of platina, and the zinc applied; as the nitrate, acetate, &c. The insoluble compounds, as the oxides, patent yellow, the carbonate, sulphate, tartrate, &c. require previous mixture with diluted aquafortis or muriatic acid. The lead thus reduced is usually of a dark grey colour, and exhibits little lustre, unless pressed, as by the blade of a knife, when its metallic lustre becomes quite apparent. It feels quite soft as the zinc is drawn over it. Its cohesion to the platina is generally so feeble that most of it may be easily removed by a cloth

or the finger. It is also presently dissolved by the aquafortis and muriatic acid of commerce, and by strong nitric acid.

I coated one side of a platina spatula with gold, mixed a drop of acetate of lead with one of diluted aquafortis, and applied the zinc; when the lead being reduced, the surface was washed, and by slight friction most of the lead was removed; but there remained sufficient to show the colour and lustre of the metal. It dissolved by diluted aqua regia, leaving the surface of gold apparently unaltered.

Compounds of copper.—The soluble salts of copper, as the perchloride, permuriate, sulphate, nitrate, &c. when dissolved in water, require only to be placed on platina, and the zinc applied, when the copper will be presently reduced and cover the platina. The compounds insoluble in water, as the proto-chloride, proto-muriate, the carbonate, oxides, &c. when previously mixed with a little muriatic acid, or diluted aquafortis, are also readily reduced. The last acid seems best adapted in the case of different pigments of copper, as Olympian and Scheele's greens, common verdigris, refiner's verditer, &c.

The copper reduced on platina, by zinc, usually exhibits the colour and lustre of the metal. Its surface is in some cases blackish, and requires gentle rubbing to show the metallic appearance. It has, in general, greater lustre, in cases when diluted aquafortis is used, and the acid only in slight excess when the zinc is applied. It is soluble in muriatic acid, but more readily in strong or diluted aquafortis.

4. *Experimental evidences that the accuracy of the Electro-chemical method of detecting metals is not impaired by the presence of mixed vegetable and animal substances, when applied to the detection of metallic poisons; particularly to the common compounds of Arsenic, Mercury, Lead, and Copper, in such mixtures.*

I have ascertained that the electro-chemical method is competent to the detection of very minute quantities of the different metals, when their compounds are mixed with vegetable and animal substances. But the object of this section is to notice briefly a number of miscellaneous experiments on the principal well known poisonous compounds of arsenic, mercury, lead, and copper, when mingled with organic fluids and solids, &c.

Arsenious acid.—I mixed a small quantity of solid arsenious acid in powder, with the following substances, separately, viz. wheaten flour and its paste; bread; cake with currants and caraway seeds; starch in powder, and in solution; rice in powder, and boiled with water to a pulp; potatoes boiled, and roasted; sugar in powder, and in syrup; vinegar, and raspberry vinegar; port and sherry wines; gruel, thick and thin, with sugar and milk, also with sugar and wine; milk and cream; white and yolk of egg, both fluid and coagulated; gelatine (isinglass) in solution; bile discharged from the stomach, mixed with saliva, and watery fluid.

In a number of those instances, it was only necessary to put a little of the mixture into the platina spoon, incorporate a few drops of muriatic acid with it, and apply a slip of zinc, when the arsenic was readily precipitated on the platina. In cases when the arsenious acid existed only in very minute quantity, or when the water present was considerable, the fluid was concentrated by boiling, or evaporated nearly to dryness, previous to the application of the zinc, when the arsenic was in like manner deposited.

Arsenious acid was also mixed with butter, lard, and oils; some of the respective mixtures being boiled a short time in the platina spoon with muriatic acid, or in solution of fixed alkali, then muriatic acid added in excess, and the zinc applied, when the arsenic readily appeared.

A few drops of an aqueous solution of arsenious acid were mixed with some sheep's blood; muriatic acid was added; the whole formed a deep wine yellow solution. A drop or two being put on the platina foil, and the zinc applied, coagulation took place, and the arsenic precipitated. Ox-bile being treated in a similar manner afforded analogous results.

In one instance, I mixed five grains of solid arsenious acid in about half a pint of tea sweetened with sugar and milk. It was kept hot for some minutes, and occasionally stirred. A little of the tea being put into the platina spoon, and muriatic acid added, the application of the zinc produced no immediate effect. But on boiling the spoonful of tea nearly to dryness, then adding two or three drops of muriatic acid to the residual brown substance, and agitating, a yellow solution was formed. A slip of zinc being now applied, a white coagulum, which soon changed to brown, appeared, and the arsenic was copiously precipitated on the spoon.

In another experiment, a small quantity of arsenious acid, in powder, was mixed with hot tea. Some hours after it had cooled, it was filtered, and on being treated as in the preceding experiment, the arsenic appeared in a similar manner. Coffee, on being exposed to like treatment, afforded analogous results.

I mixed five grains of pulverized arsenious acid in a small basin of warm and rather thick pea-soup, having fragments of the fibrous part of beef diffused through it. The platina spoonful of it was boiled nearly to dryness, several drops of muriatic acid were added, and by agitation most of the solid matter was dissolved, forming a thickish fluid. The zinc being applied, a white coagulum changing to brown appeared, and the arsenic soon covered the surface of the spoon. Similar results were obtained when giblet soup was treated in like manner.

Corrosive sublimate.—A few drops of a solution of corrosive sublimate in water, were well mixed with a solution of gelatine (isinglass): a drop or two of the mixed fluids being placed on the platina spatula, and the zinc applied, the mercury presently precipitated; and this effect was more readily produced when a little diluted muriatic acid was previously added to the mixture. When a solution of nutgalls was added to the mixed gelatine and corrosive sublimate, also when corrosive sublimate was added to yolk of egg, the results, by similar treatment, were precisely analogous.

A small quantity of solution of corrosive sublimate was put into fluid albumen (white of egg). To some of the precipitate in a platina crucible, a little diluted muriatic acid was added, and the zinc applied; the albumen coagulated, and in about a minute some mercury was reduced in the crucible, and by continuing the experiment for a few minutes the quantity increased considerably. A similar experiment being repeated in a platina gilt crucible, in a short time the gold partially assumed a dark blueish grey colour. The mercury from both experiments was collected by sublimation on glass, as detailed before.

When solution of corrosive sublimate was mixed with flour into a soft paste, put into the platina crucible, and diluted muriatic acid added, in the course of a few minutes after the zinc was applied, similar results as in the foregoing instances, were obtained.

Corrosive sublimate in powder was mixed with butter, and diluted muriatic acid was incorporated with the mixture; a little of the same being placed on

the platina spatula, and the zinc applied to the fluid, the mercury was presently reduced. The experiment was successfully repeated with diluted aquafortis, instead of muriatic acid.

Solution of corrosive sublimate was respectively mixed with saliva, sheep's blood, and ox bile; the zinc being applied to each, on platina, the mercury was soon reduced.

A few drops of solution of corrosive sublimate were mixed with some human bile, and watery fluid discharged from the stomach, and rather more than half its bulk of muriatic acid was added; a small teaspoonful of the mixed fluids was put into the platina crucible, and the zinc applied for about two minutes, when the surface of the crucible had acquired a dark grey colour; on being heated, metallic mercury was obtained. The experiment was repeated, with similar results, in a platina gilt crucible.

A solution of corrosive sublimate, in small quantity, was mixed with milk; muriatic acid, equal to about one half the milk, was added. Some of the mixed fluids was put into the platina crucible, and the zinc being applied a short time, a grey surface of mercury appeared on the platina. A similar experiment was made in the platina gilt crucible, which soon exhibited a partially whitened surface. From both experiments mercury was sublimed.

A few drops of solution of corrosive sublimate were mixed in a cup of tea sweetened with sugar and cream. About a teaspoonful was put into the platina crucible, with nearly an equal bulk of muriatic acid. A slip of zinc being applied for about a minute, a greyish substance was found at the bottom of the crucible, which, when washed, dried, and heated, afforded metallic mercury. These experiments were repeated with success, both in the platina crucible and in the platina gilt crucible; and precisely similar results were obtained with coffee, under like treatment.

It is proper to remark, that the experiments noticed in this section, were, in general, carried on only for a few minutes, so that the decompositions were mostly partial, and probably in no instance complete. This statement is confirmed by facts, for I repeatedly found that after a given effect had taken place in the platina crucible in a few minutes, a still further effect was soon produced, by transferring the same materials to the platina gilt crucible.

When a solution of corrosive sublimate (in small quantity) was added to some sherry or port wine in a platina crucible, and the zinc applied, a beautiful effect gradually took place. In the course of about a quarter of an hour, calomel was formed, and deposited at the bottom of the crucible in successive circles, which extended beyond each other, and differed slightly in their shades of colour, being alternately lighter and darker. The zinc became brittle at its point of contact, from amalgamation. These experiments were repeated in the platina gilt crucible with similar results, part of the gold was whitened, and calomel and mercury were afterwards sublimed from both crucibles.

With raspberry vinegar and solution of corrosive sublimate, the results appeared to be similar to those produced with the wines.

Sugar of lead, acetate of lead.—A solution of sugar of lead, in small quantity, was separately mixed with flour into a thin paste, with saliva, port and sherry wines, raspberry vinegar, and yolk of egg. A little of each mixture was put on the platina spatula, and the zinc applied, when the lead was soon reduced on the platina.

A little of the solid compound of albumen (white of egg) and sugar of lead was placed on the spatula, and mixed with a drop or two of diluted aquafortis; the zinc being applied, the lead was reduced to a dark grey colour; by gently rubbing it with the finger the lustre of the metal became apparent. The results were analogous when sugar of lead was mixed with sheep's blood and ox bile, and exposed to similar treatment.

A mixture of gelatine (isinglass) and sugar of lead was put on the spatula, and the zinc applied; the lead was slowly reduced: the effect was more rapid when a drop of diluted aquafortis or muriatic acid was previously added. The results were precisely similar when a solution of nutgalls was added to the mixed solutions of gelatine and sugar of lead. The yellow soft solid readily yielded metallic lead on platina by means of zinc.

A small quantity of sugar of lead was put into some tea sweetened with sugar and cream; some of the yellow matter produced being put on the spatula, and the zinc applied, the lead was slowly deposited. The results were similar when sugar of lead was mixed with milk and with coffee. The addition of a little diluted aquafortis appears, in a number of cases, more readily to

reduce the lead, but as a slight excess of acid will redissolve the lead (partially or wholly), a longer contact of the zinc is necessary to neutralize the acid, and render the lead permanent.

Sulphate of copper, blue vitriol.—A small quantity of sulphate of copper in solution was added to the following substances, viz. flour forming a soft paste, milk, tea and coffee sweetened with sugar and cream, gelatine (isinglass) in solution, and with nutgalls, albumen (white of egg) and yolk of egg, saliva, sheep's blood, and ox-bile. On a little of each mixture being placed on the platina spatula, and the zinc applied, the copper was presently reduced on the platina. In some cases, the surface was of a blackish colour, but its proper colour and lustre became apparent on rubbing it. The addition of a little diluted aquafortis or muriatic acid to the mixtures, usually facilitated the reduction of the copper.

A little sulphate of copper in powder was mixed with butter and lard, placed on the spatula, and a drop of diluted aquafortis incorporated with them. On the zinc being applied, the copper was readily reduced.

5. *Some general Remarks, and Conclusion.*

The experiments detailed in this paper seem to prove, that the common compounds of arsenic, mercury, lead, and copper, may be readily reduced to the metallic state, on platina, by the electro-chemical method described; and that this method is also competent to the detection of those metals, in cases when their compounds are mixed in very small quantity with vegetable and animal substances. It may, I think, be further deduced, that the method is applicable in cases when those compounds exist in the most complicated mixtures of organic substances. For, though I have had no opportunity of applying it to the contents and tissues of the stomach in instances of poisoning, yet its efficacy in such cases can scarcely be doubted, as no animal or vegetable substances can, I apprehend, resist the action of the mineral acids, which are almost indispensably necessary to the success of the method. In some instances, where the common tests will not act at all, and in others, where they only act fallaciously, the electro-chemical method will, in general, be found to act with certainty.

The general results appear to be strictly electro-chemical, or to arise from

the union of a chemical and an electrical action, and seem to be comprehended in the principle developed by Sir HUMPHRY DAVY relating to the changes and transitions by electricity, viz. that "the metals are attracted by negatively electrified metallic surfaces, and repelled by positively electrified metallic surfaces, and these attractive and repulsive forces are sufficiently energetic to destroy the usual operation of elective affinity *."

If the decompositions which I have noticed were effected by the mere contact of the zinc with acid solutions of the metals, there would, indeed, be reductions, but no depositions on the platina. The electrical action arising from the contact of the two metals, in cases when small slips of zinc and platina were employed, was quite sufficient to destroy or suspend the usual operation of chemical affinity. Thus, in numerous instances of the decompositions of metallic compounds, the platina remained covered with the reduced metal during the contact of the zinc; but on removing the zinc, the coating was readily redissolved by the slight excess of acid present. The transition of the reduced metal to the surface of the platina, is the effect that gives to the method a beauty, simplicity, and delicacy, exceeding, in my opinion, all other modes of detecting the metals already referred to, together with a number of others which remain to be noticed.

As this paper has already exceeded the limits at first intended, I shall defer noticing a number of useful applications of the facts, until I shall have the honour of submitting to the Society the remaining part of the subject.

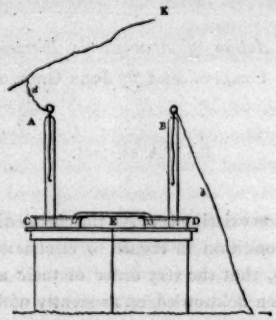
* Phil. Trans. 1807.

IX. *On the Chemical Action of Atmospheric Electricity.* *By* Alexander Barry, *Esq. F.L.S. Communicated by* John George Children, *Esq. Sec. R.S., &c. &c.*

Read Feb. 24, 1831.

THE intimate connection existing between the chemical constitution of various substances and their condition in regard to electricity, has now been long known. So true is this, that the very order of their affinities has also, for a considerable period, been acknowledged as greatly within the control of this agent. With this object in view, the electrical influence, as exhibited in the ordinary way, as well as by the Voltaic battery, has successively confirmed its results in the experiments of Nicholson, Cavendish, Sir H. Davy, Children, and the French chemists. This being the case, it is not my purpose to advert to any new source of electrical accumulation, but to describe what appears to me as the link connecting the researches of Dr. Franklin with the electro-chemical theory of Sir H. Davy. With this view, in August 1824, I elevated the kite in an atmosphere favourable to the exhibition of its phenomena. It was raised from an apparatus firmly fixed in the earth, and was insulated by a glass pillar. The usual shocks were felt on touching the string, which simple fact I am induced to mention from the circumstance of no electrometer having been employed. The portion of string let out with a double gilt thread passed through it, was about five hundred yards. I then made the connection shown in the accompanying sketch, where the straight glass tubes A B, having platina wires passed from above half way down their axes, and standing in their respective glass cups C D, were filled with a solution of sulphate of soda coloured with syrup of violets, connected also with each other by the bent glass tube E, likewise filled with the above solution in the usual manner. A portion of gilt thread *d* was then brought from the tube at A and united to the kite-string K, whilst a similar thread *b* was carried from B to the earth. Bub-

bles of hydrogen in A and of oxygen in B, soon appeared. In about ten minutes, the blue liquid in A became green from the separation of the soda,

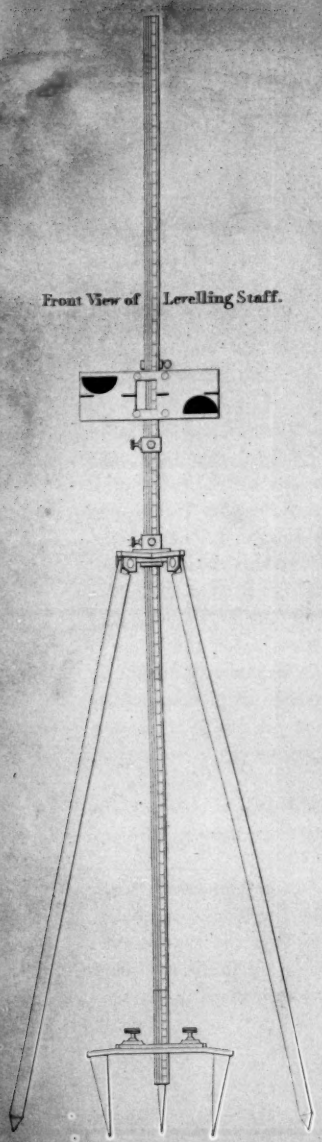


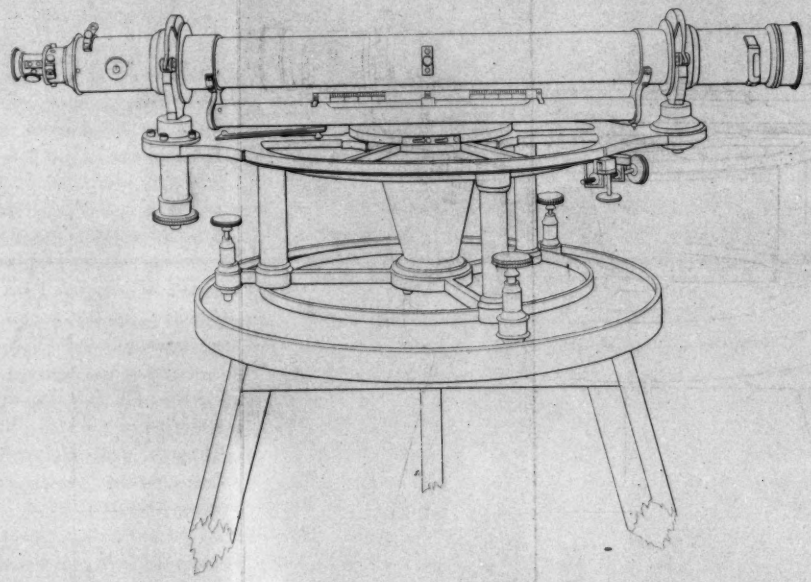
whilst the sulphuric acid, by passing to the pole in the tube B, changed its contents, as usual, red. The experiment was then discontinued. In having the honour to communicate the result to the Royal Society, I must remark, that it was my intention to have pursued this simple application of electricity, in the expectation of arriving at more extended and important conclusions, which, however, numerous circumstances have hitherto prevented me from accomplishing.

Fig. 1.

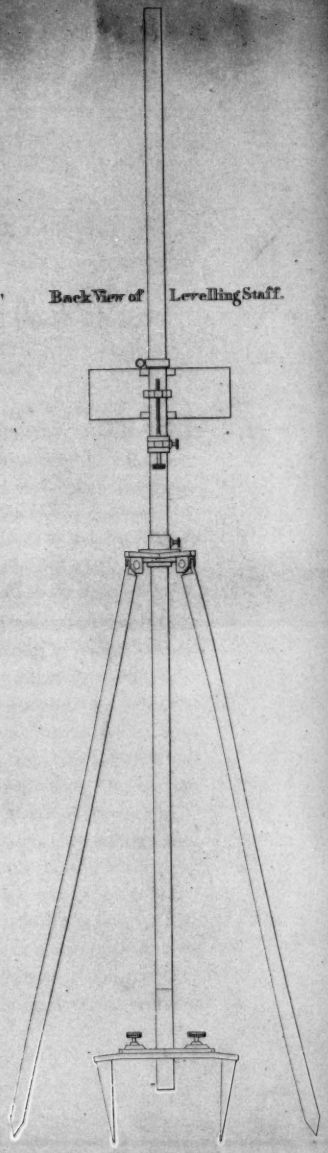
GENERAL VIEW OF THE LEVELLING INSTRUMENT.

Front View of Levelling Staff.





Back View of Levelling Staff.



X. *An Account of Operations carried on for ascertaining the Difference of Level between the River Thames at London Bridge and the Sea; and also for determining the Height above the Level of the Sea, &c. of intermediate Points passed over between Sheerness and London Bridge.* By John Augustus Lloyd, *Esq. F.R.S. F.R.G.S. & F.S.A.*

Read March 3, 1831.

In February 1830, at the suggestion of the Royal Society, I had the honour to receive directions from the Lords Commissioners of the Admiralty to make such observations as I might consider necessary, to ascertain the difference, if any, between the level of the waters at certain points on the river Thames, and the mean level of the sea near Sheerness, as well as the height of different intermediate points above the sea, such as Gravesend, Greenwich Observatory, &c.

Having found, while employed in the Isthmus of Darien, how inadequate the present levelling instruments were to obtain very accurate results, and being desirous of conducting the interesting observations, I now had orders to make, with the most scrupulous exactness, I thought it necessary, in the first instance, to bestow some attention to the improvement of the instruments required to be used, endeavouring to combine superior steadiness and motion in azimuth, more delicacy in the level itself, more permanency in its position, and greater power in the telescope.

After several trials, and by the assistance of Mr. Cary, (to whom I am indebted for many valuable suggestions,) I determined upon having an instrument made exactly after the accompanying Plate.

Fig. 1. is a perspective view of the instrument. It is supported by three foot-screws, similar to those used in the best altitude and azimuth circles, with thirty threads to an inch, and serving to place the instrument horizontal.

The stand is formed by two plates of brass, which are firmly connected together by three pillars. To the limb that carries the telescope, in very sub-

stantial Ys, is fixed the inner conical centre, three inches in diameter and eight in length, ground to move perfectly smooth in the hollow conical centre fixed to the stand.

This limb projects over the upper part of the frame about one inch, and is bevelled at the edge, in order to slide in a groove attached in two pieces to a clamp, screwed underneath the upper part of the frame, and which, by means of a tangent screw, gives a slow motion in azimuth to the limb. This limb, having two solid projections, allows the Ys being at some distance over the periphery of the circle, by which means the supports to the telescope are nearer its extremities. One of these Ys has a short vertical motion, with a pushing screw at the side, which fixes it at any height required. The telescope itself is of thirty inches focal length, and magnifies about twenty-two times, having two thick gun-metal collars, by which the telescope rests on and is turned in the Ys. These were quickly worn by the continued friction from the supporting part of the Ys, and would have caused an error in using the instrument; I therefore had small thick steel plates, highly polished and hardened, dovetailed into each supporting part of the Ys. The friction on the collars was now transferred to the pieces of steel; and although they likewise continually wear, it does not affect the correctness of the instrument in any other way, than in altering the adjustment of one of the Ys. (See page 20.)

At the eye end of the telescope is an adjustment with a rack and pinion, for distinct vision, and another by the same means to regulate the distance of the eye from the wires: this I found to be indispensable, as the eye occasionally becomes fatigued, and requires a different focus to view the wires distinctly.

The wires themselves are adjusted as usual, by a sliding piece for azimuth, and two pushing screws for vertical motion.

In the centre of the telescope there are two orifices opposite one another; the one to receive a small lamp, and the other admitting a spindle and speculum at the end, ground to an angle of 45°, which, by reflection, illuminates the wires for night observation. The speculum being made to turn, the quantity of reflected light may be regulated at pleasure.

As the glass would not admit of distinct vision for objects at a less distance than one hundred feet, and it being necessary to use the instrument, at times,

not more than three feet distance from the station-staff, several lenses were made of different focal distances, as sixty, thirty, fifteen, eight feet, and thirty inches; which, being applied to the object end of the telescope, converts it, in fact, to a microscope.

To the lower part of the telescope, within the collars, are affixed the cocks, into which fits the tube protecting the level: one of these cocks gives a vertical, and the other a horizontal motion to the tube, in order to place the level parallel to the axis of the telescope.

When the instrument was first made, these motions were effected by means of endless screws; but I found it so difficult (almost impossible) to keep a delicate level in adjustment by this mode, that I substituted the old fashion of capstan-headed pushing screws.

To the upper part of the telescope are attached (outside either points of support) cocks* or braces, carrying a swinging level, having, as well as the cocks, separate adjustments.

This additional level was intended as a check to the lower level, and to detect any occasional variation in the figure of the tube itself.

The glass bubbles themselves were placed in the tubes at first with paper, and wedged at either end with small pieces of wood; but the wedges are liable to distort the bubble itself, and after some time get loose in the tube; and the level alters in its position, and is never to be depended on. I found it better to push the level into its proper place in the tube, not tightly, and with paper underneath, taking care that the paper touches the middle part, and then filling up the two ends with plaster of Paris.

To the upper part of the telescope, near the eye-piece, is fixed a small level, adjusted to the horizontal wire of the telescope, and by the assistance of which the same surface of the large level is used at each observation. There is also another small level at the other end of the telescope, to adjust the vertical wire by, but of less use than the former.

On the upper limb, near one of the Ys, is fixed a small thermometer, with the bubble inclining downwards, at an angle of about 10°, the use of which will be explained hereafter.

The large stand of this instrument is made with a solid top of African oak.

* These cocks are not shown in the Plate.

The legs, which are very strong, are iron-shod, and braced at the bottom, about six inches from the shoeing, with three thick iron rods, rendering the whole steady, and affording the men a purchase to press down with their feet, and rough level the stand ready for the reception of the instrument.

Adjustments.

The above instrument requires several adjustments, which I shall endeavour to describe in the order they are made.

First, To make the lower level parallel to the axis of the telescope:

Place the telescope directly over one of the foot-screws, and clamp it; then bring the bubble of the level, by means of the foot-screw, to the same division on either side the graduated scale affixed to and over the level, taking care that the bubble of the little level at the eye end of the tube (and at right angles to the large level) is in the centre; then reverse the telescope in its collars, observing if the bubble reaches to the same division, and correct one half of that number by the pushing screws on the level itself, and the other half by the foot-screw: this must be repeated until the bubble remains in the same spot.

Second, To place the large level in the same vertical as the axis of the telescope:

Move the telescope in its collars until the level is brought considerably to one side, and observe if the bubble still remains at the same division; if not, move the side pushing screws on the level, until the bubble has returned to its proper place; move the telescope again as much to the other side, and observe if the bubble comes to the same division; if not, it must be re-adjusted, until it is as near as the accuracy of the grinding of the level will allow.

These two adjustments are naturally dependent on one another, therefore they must be both examined, until no alteration in the bubble can be perceived.

Note.—As the collars may wear a little hollow in time, care must be taken that one particular shoulder of the two collars rests against the Y when reversed, in order to use the same point of support.

Third, To place the axis of the telescope parallel to the plane of the instrument:

Loosen now the clamp which confines the telescope over the foot-screw, and adjust the bubble exactly to the centre; bring the telescope and limb half round on its conical centre, and observe the bubble; half the number of divisions, in error on the scale, must be corrected by the foot-screw, and the other half by the vertical motion to the Y, securing it, when it is sufficiently adjusted, by the little side screw.

Fourth, To place the vertical and horizontal wires at right angles, and to connect them with the little levels on the telescope:

Place one of the station-staves about five feet from the ground in a horizontal position (by means of a small hand level and two nails); from the centre thereof, suspend a white plumb-line; adjust the vertical wire to the horizontal one, by loosening the two screws, which admit of its moving diagonally, and making them coincide with the surface of the staff and the plumb-line; then by loosening the screws of the small level at the eye end, move it until the bubble rests in the centre, reverse the position of the wires, making the vertical horizontal, and adjust in the same manner the other little level.

Fifth, To make the hanging level parallel to the plane of the instrument:

Adjust by the foot-screw, until the bubble of the lower level is in its position; then observe the variation of the riding level, and alter it one half the error by the vertical screw on the cock, and the other half by the pushing screws on the riding level itself.

Or, By the foot-screw, bring the bubble of the hanging level in the centre; then reverse it on the cocks; one half the difference is to be altered by means of the adjustment on the level itself, and one half by the foot-screw: now place the lower level perfectly horizontal, and by the vertical screw on the cock, bring the bubble of the hanging level to correspond.

The usual adjustment for collimation is here purposely omitted. The eye tube of the telescope altering continually in its position, renders it most difficult to make this adjustment correctly; for although it may be found in perfect adjustment for a distant object, when directed to a near one there will be a considerable error, which would affect the results in levelling, if the station-staves were not equidistant from the instrument.

To avoid this inconvenience therefore, as well as to avert the difficulty of placing the extra lenses for short distances, so as not to alter the line of colli-

mation, I permitted the wires to be some distance from the axis of the telescope; and, in levelling, a mean of four or six observations were taken, with the telescope turned half round in its collars at each observation.

Note.—I take this opportunity of mentioning a substitute I have occasionally used for wire or cobweb with success, viz. asbestus; very fine fibres of which can be obtained by being thrown into hot water, when it easily divides. These fibres are tough enough to be placed with ease on the diaphragm, and have the advantage of being opaque.

Being now in possession of an instrument equal to perform the most delicate observations, my next object was to make some improvement in the station-staves, so that they might point out as minute a quantity as the instrument could detect a difference of. This was something difficult, without encroaching on the portability and quick application of the staff to its use.—The following is a description of the staves I used, a Plate of which is given.

The staff itself is a rod of six feet six inches in length, of solid seasoned mahogany on the face of which is let-in a slip of brass, riveted at intervals from end to end; by the side of this is also a slip of holly, fixed in the same manner. The divisions are laid down in feet and tenths, on the holly; and in feet, tenths and hundredths, on the brass. The lower part of the staff is fitted into a square tube of brass about eight inches long, four inches of which are occupied by the staff, and the remainder filled up with lead.

The vane is a plane of seasoned holly, with two semicircles of stained ivory let into the face of it (see Plate). It is fixed by screws to a brass box with tightening springs on the staff itself: there is another small slide on the staff, having two clamping screws, and a long tangent screw, which is attached to the box of the vane by a female screw; the last gives a slow vertical motion to the vane.

On the top of this vane, and at right angles to one another, are two small spirit-levels, mounted in brass; affixed to the vane, in a square hole in the centre, and levelled on one side, is a small brass vernier, the edge of which slides on the divisions of the staff reading to the 1000dth, and, by practice, to the 10,000dth of a foot.

The station-staff thus described is in itself complete; but for accurate observation it requires to be immoveable on the picket: a three-legged support

is therefore added, having a box and ring with double compass gimbles, and a horizontal motion; into this the staff slides.

There is also a small brass tripod with iron legs, having a hole in the centre plate, over which another small plate slides, fixing to the tripod by two clamping screws; this is used to confine the bottom of the station-staff over the picket.

Adjustment.

The only adjustment required to this instrument, is to enable it to be placed vertically over the picket.

Slide the staff into the piece carrying the gimbles; suspend it with the gimbles as nearly as possible at right angles, and as distant as conveniently may be from the ground. When the staff does not oscillate, observe if the two bubbles on the vane are correct; if not, by means of the small screws at either end raise or depress them, until the bubble remains in the centre.

Having now the means of placing the staff immoveably and vertically over a spot, it is to be accomplished as follows: Fix the support that carries the staff, as nearly as possible over the required spot; pushing the legs into the ground, place the small tripod as shown in the Plate, unclamp the milled heads, and pass the staff through the gimbles of the support into the plate of the tripod. As this plate will move in any direction horizontally, the staff is to be adjusted, until perfectly upright, by means of the levels on the vane; then clamp the plate firmly, and push the staff down to the head of the picket, and turn it until it is directly in front of the level.

In March 1830, having every thing prepared, I departed for Sheerness, determining to commence my observations at that point.

As part of the main object of my commission was to ascertain the height of different places above the level of the sea, it was necessary to endeavour to determine that point with accuracy. From the observations made from time to time, at the caisson at Sheerness Dock Yard, of high and low water, I could only obtain this point within a certain degree of accuracy. I therefore determined to take advantage of the permission granted me by the Admiralty, to commence the erection of a tide-gauge at the Dock Yard. Accordingly, after having a model made on a principle that I hoped would be the most simple

and yet the most accurate, I examined the whole of the front of the Dock Yard, and selected the corner of the boat basin adjoining the ordnance basin, as the only spot indeed at all eligible for the erection of a tide-gauge. (The spot is shown on the plan of the Yard.)

By the friendly and prompt assistance of His Majesty's Commissioner at the Dock Yard (J. Lewis, Esq.), I was enabled to have my tide-gauge quickly and substantially erected, under the excellent superintendence of Mr. Mitchell, the master millwright at the Yard.

Description of the Tide-Gauge.

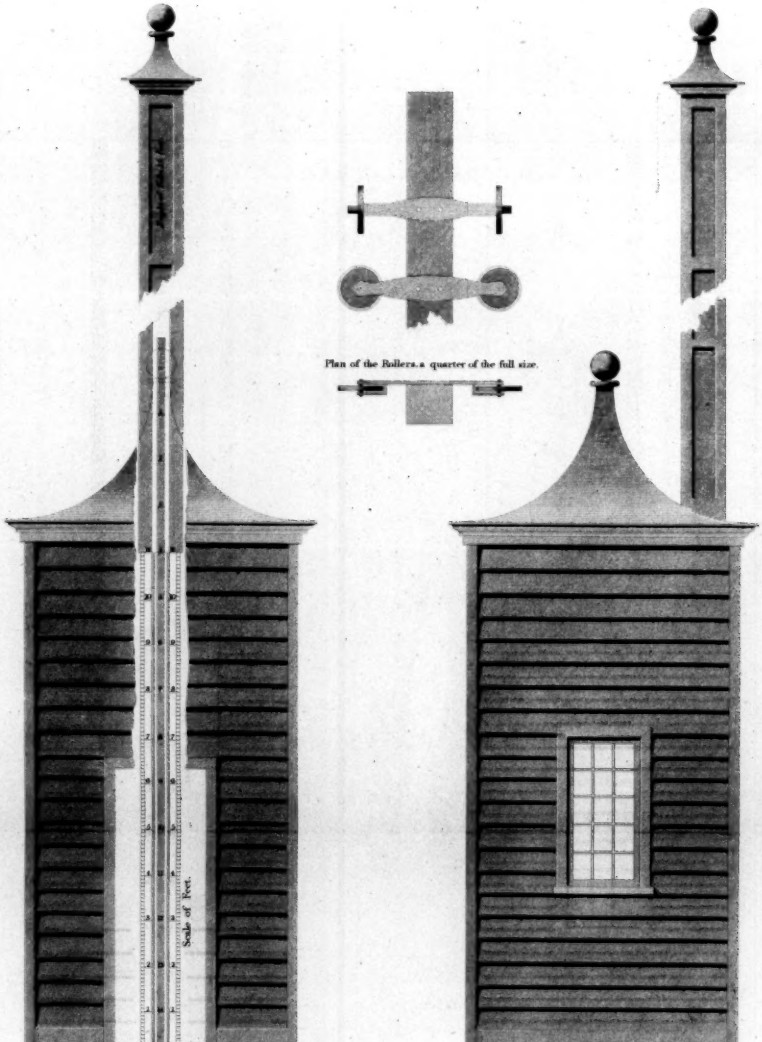
The model is on a scale of three quarters of an inch to a foot.* The flat board on which the house stands, represents the Wharf at Sheerness; it is made long at the back, and balanced so as to be placed on a table, to show the trunk in the actual position of the gauge. The top or cap to the covering of the staff is left unfixed, in order to be taken off, and allow the house to be lifted over it, and show the gauge alone.

The slide rods in the model are of iron wire, and out of proportion; but no smaller was at that time to be found in the Dock Yard; in the original they are ¼ copper bolts.

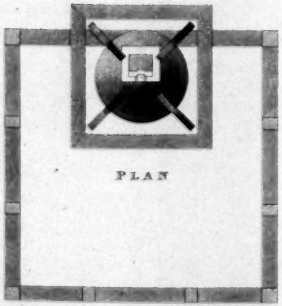
The slides or pointers have springs to tighten them to the rods, but they are too minute to act in the model. The lower end of the tube in the model is nearly filled up with wood, in order to secure it to the trunk, leaving only a part for the gauge-rod to pass; but in the original the tube is left open below, there being sufficient strength in the timber to allow of its being bolted to the trunk and platform. There are three small friction-wheels at the upper end of the rod, to steady it.

There is a distance of fifteen feet between the top catch or point on the rod which brings the slide down to indicate low water, and the point on the middle of the rod which takes the other slide up to indicate high water. Therefore allowing an eighteen-feet tide, the top catch will bring the slide down to four on the index, and the middle catch on the rod will raise the slide to seven on the other index, the difference of which, three added to fifteen (the length between the points), gives eighteen. Therefore it will be observed that the dif-

* The model is deposited at the Royal Society's Apartments in Somerset House.



Plan of the Rollers a quarter of the full size.



PLAN

PLAN AND ELEVATION
of the
TIDE GAUGE
at
H.M.DOCKYARD
SHEERNESS,
and at
PORTSMOUTH.

Scale ¼ of an Inch to 2 Feet.

J^s Gardner sc

ference between the slides is always to be added or subtracted, to or from the length between the points on the rod. The divisions on either index are continued from 0 feet to 11 feet, to allow for any extraordinary tides. The little brass standard in front of the tide-gauge, and about six or eight feet from it, has been connected with it; therefore the height of any particular tide above the standard is easily found, knowing the exact distance from the middle catch to the line of immersion (which is registered). For example: Let the distance from the centre catch or point, to the line of immersion be 12.444, and the face of the brass standard on a level with 0.747 of the indices; and that high water carries the pointer to 6.546 on the index, then $12.444-(6.546-0.747) = 5.151$, the height of that particular tide below the standard. Again, assuming 4.328, as pointed on the index by the upper catch for low water, then $15+12.444-(4.328-0.747) = 23.863$, the height of standard above that of low water. The difference between the two, gives 18.712 for the rise and fall of the tide.

Many interesting observations may also be made by the same instrument on the irregular rise and fall of the tides, as there is an excellent clock within the house, and I believe an index connected with a weathercock on the outside.

The trunk of the gauge is substantially fixed between large piles, driven in for the purpose, and the whole partitioned off to low-water mark, to render the gauge secure from boats or vessels*.

My next object was to select a standard mark, from whence to commence my levelling, and to form a zero point.

I made examinations of the Dock Yard in different parts, particularly that part facing the Medway. I at first fixed on the northern part of the Dock Yard wall adjoining the garrison; but upon due inquiry I found that part was not considered so good a foundation, being built on piles driven in at some distance from one another, and to no great depth. Wishing, however, to shorten the distance between my zero point and the long level I should have to take across the Medway to the Isle of Grain, and desirous of having the

* This tide-gauge is on the same principle as that mentioned by Mr. Lubbock in the Companion to the British Almanac.

standard in the vicinity of the tide-gauge, so that future observations might be easily connected with the standard, I selected a large block of granite in the southern pier of the entrance to the boat basin, the position of which is shown in the plan of the Yard. I caused a block of gun-metal (cast for the purpose), two inches and a half square and eight inches long, to be sunk in the centre of the granite, about an inch below the surface, thereby allowing a brass box and cover to be placed over the standard, to protect it from injury.

In order that there should be a sufficient number of checks to the stability of this standard mark, I caused three more to be placed in the Yard; viz. one near the southern extremity on the wall of the Dock Yard, one at the eastern side of the great basin, and one in a large block of stone resting on the brick-work of the navy well 330 feet deep.

But however satisfied I might be that these standards were sufficiently firm, I thought it advisable to seek some spot more unquestionable in its foundation than Sheerness Dock Yard. I found a place that possessed this advantage; it was a slight eminence about two miles and a half to the southward of the Dock Yard, and surrounded by a moat.

On this eminence formerly stood the old castle of Queenborough, within a short distance of the present town of Queenborough. The castle, which was in the form of a pentagon, was some years ago pulled down, but a very small part of the foundation was left. On this foundation, which is rubble and chalk, some feet under the surface a very large block of granite was placed for me, by order of the commanding officer of engineers, and into which was let one of the brass standards. The place is now covered over, but marked by a small mound of earth near it, and reference can be easily made to it if required.

Having now standard marks enough to ensure, by comparison, the knowledge of any alteration (if any should occur) in the zero point, I commenced levelling.

My first business was to ascertain the difference of the several standards in the Dock Yard above the level of the sea.

From a series of observations made at the caisson at the entrance to the great basin, in the years 1827, 1828, 1829, the mean of the tides was as follows:

Mean high water spring tide for	1827,	26.475	Low water	8.700	Mean	17.587
Do. . . . do. . . .	1828,	26.50	. do. .	8.61	do.	17.55
Do. . . . do. . . .	1829,	26.09	. do. .	8.93	do.	17.51
Means . .		26.355	. do. .	8.74	do.	17.549
High water neap tides . . .	1827,	22.56	. do. .	11.12	do.	16.84
Do. . . . do. . . .	1828,	22.69	. do. .	11.44	do.	17.06
Do. . . . do. . . .	1829,	22.72	. do. .	11.45	do.	17.08
Means . .		22.656	. do. .	11.336	do.	16.993

Mean for the three years.

Spring tide high water,	26.355	Spring low .	8.74	Mean level	17.549
Neap tide do. . .	22.656	Neap low .	11.336	. do. .	16.993
Means . .	24.50		10.03		17.27

As there are many blanks in the three years' observations that these were taken from, I have also selected the most perfect year (1827), and taken the mean of all the tides for each month in the year.

The following is the

Summary.

January, Mean high water	24.78	Mean low	10.20	Mean level	17.49
February . . do. . .	24.08	. do. .	10.00	. do. .	17.04
March . . . do. . .	24.70	. do. .	9.60	. do. .	17.15
April . . . . do. . .	24.25	. do. .	9.90	. do. .	17.07
May . . . . do. . .	24.35	. do. .	10.35	. do. .	17.35
June . . . . do. . .	24.31	. do. .	10.29	. do. .	17.20
July . . . . do. . .	24.39	. do. .	10.16	. do. .	17.27
August . . . do. . .	24.39	. do. .	10.26	. do. .	17.32
September . . do. . .	24.29	. do. .	10.00	. do. .	17.14
October . . do. . .	24.79	. do. .	9.66	. do. .	17.22
November . . do. . .	24.65	. do. .	9.90	. do. .	17.27
December . . do. . .	24.53	. do. .	9.62	. do. .	17.27
Means . .	24.46		9.995		17.23

These results agree so very nearly, that they may be safely taken as correct. I have given them at length, to afford the data for finding the true level of the sea. It will be seen that the mean level taken from spring tides differs 0.556 from the mean level deduced from neap tides. I shall, however, assume the mean level as 17.27, differing only 0.04 from the same of the whole year's observations of 1827.

At the north-western end of the caisson I caused a small cross (+) to be cut in the granite, which corresponded exactly with the XXXI feet of the indices cut in the masonry at the side of the caisson. This cross is situated as follows, with regard to the standard marks in the yard. (See page 3 of Levelling Book.)

The northern standard, which is my zero point, is 0.5789 below the cross. The southern standard at the other end of the Dock Yard wall is 0.0259 above the cross; therefore $0.0259 + 0.5789 = 0.6048$ gives the height of the south standard above the northern one, although they are set equally beneath the surface of the granite.

The standard at the eastern edge of the great basin is 0.6735 above the northern, and $0.6735 - 0.6048 = 0.0687$ above the southern standard, and $0.6735 - 0.5789 = 0.0946$ above the +.

The standard at the navy well is 6.0656 below eastern standard.
5.9968 below south standard.
5.3920 below north standard.
and *5.9710 below the cross.

Now, the mean level of the sea being 17.27 feet, the + or XXXI feet of the indices is 13.73 above the mean level of the sea, and 4.645 above the mean spring tide high-water mark. And

North standard or zero is	$13.73 - 0.5789 = 13.1511$	above the level of the sea, and	4.0661	above spring tide high-water mark.
South do.	$13.73 + 0.0259 = 13.7559$	do.	4.6709	do.
East do.	$13.73 + 0.0946 = 13.8246$	do.	4.7396	do.
Navy well	$13.73 - 5.9710 = 7.7590$	do.	4.3260	below do.
Tide-gauge standard	$13.73 - 0.0946 = 13.6354$	do.	4.5504	above do.

* In page 4 of Levellings there will be observed a difference of 0.0001 in the two sets of levels, the first set taken in March, the last in June.

Having now completed this part of my observations, I levelled from the Dock Yard to Queenborough:—the result of the operation will be seen in pages 1 and 2 of the Levelling Book. And it will be there observed, that between the levels and proofs there is a difference (in a distance of nearly three miles) of 0.0269 about $\frac{3}{8}$ths of an inch. In the commencement of my commission I had a new instrument, and many little difficulties to overcome. In part of the Queenborough observations, I made use of another level which I had from Troughton, on a different principle to the one by Carey; and in page 3 of the rough Levelling Book, an observation is there made on the difficulty I had in getting the level to adjust, from the fault I have mentioned in my remarks, of the bubble having been placed in its tube with only a little paper, and becoming loose.

I used also my large instrument with a level adapted to it, made by Riechenbach, which was lent me by Captain Sabine. It was of a large diameter, and ground most accurately, but not hermetically sealed, having two plates of glass ground in to the ends. It was exceedingly delicate, but at any considerable motion, the bubble would break into many globules, requiring a long time to collect again. The second day I used it in the marshes the day was hot, and I had not at that time the means of shading it from the sun; the bubble, while observing, suddenly contracted, and almost immediately disappeared. I threw some water on the instrument, and the bubble gradually appeared again, but much lengthened. The next day I placed the level in the sun with a thermometer; when at 68° the end was forced out, and I found it contained ether, which appears to be quite unfit for a level in a high temperature. For future observations I replaced it by one of Cary's make.

The mean of the observations give as follows:—Queenborough standard is higher than southern standard 9.9981, and 10.6029 higher than northern standard; and 14.6690 above spring tide high-water mark, and 23.7540 above the mean level of the sea.

Having driven a large picket into the embankment at the Isle of Grain, at a point the shortest distance I could obtain from the north standard (viz. about 5000 feet), across the Medway, I now commenced the observations necessary to ascertain the difference of level between the zero or north standard, and the picket at the Isle of Grain. One instrument was placed near

the standard mark, and the other on the Isle of Grain, about fifty feet from the picket.

Between the high-water mark at the Isle of Grain and the standard at Sheerness, there is a long ledge or bank of mud, extending nearly half a mile, and quite dry at low water. At these times I found it very difficult to make observations, and together with the wind occasionally interfering, and the interruption from the sun's rays, which caused a great vibration in the atmosphere, I spent day after day uselessly waiting at the instrument. I was, however, more successful afterwards, by taking advantage of the first hour or two of day-light, which was more favourable for observation, as the wind had generally lulled, and the air was in a quiescent state.

After the instruments were in perfect adjustment, an observation was made at each side of the river alternately; first by myself at Sheerness, and then by my assistant at Grain. A mean of a considerable number was taken, the difference of which, after the curvature was subtracted from each, would give the refraction, provided the instruments were in good adjustment, and the observations correctly taken. (See page 5 of Levellings.)

Not being satisfied with the observations made at my first visit to Sheerness, I returned at a subsequent period, and made a great number most satisfactorily, from which the best were selected, and a mean taken.

As there was not a fit place for a standard mark near the margin of the river Medway, I caused a block of granite, about 900lbs. weight, to be placed on the foundation of the old church wall (the church was once much larger) of St. James, near the porch-door, into which block was sunk one of the brass standards. (See page 6 and 8.)

The face of this standard is 28.7454 feet above the north standard mark; 32.8115 above spring tide high-water mark, and 41.8965 above the mean level of the sea.

From St. James I almost immediately came into the marshes, levelling in a direct line towards Yantlet Creek. In consulting the Ordnance map of Kent, there appear but two or three ditches in the marshes; but in a distance of less than three miles I passed thirty-three dykes and ditches, from fifteen to twenty-five feet broad, and four or five deep, over which there is no mode of passing, but a bridge perhaps a mile distant.

It was desirable therefore to find some means of conveying my instruments, men and apparatus over, without losing so much time by going round. The men first used leaping-poles; and a case for the instrument was constructed water-tight, so as to swim over the ditches. But from the sides of the ditches being steep, the instrument was constantly subject to heavy blows in being taken out of the water; I therefore adopted another mode. On our arrival at a broad ditch, two leaping-poles, about eighteen feet in length, with cross pieces at the bottom to prevent their sliding too far in the mud, were connected together at the top by passing the eye of a rope just over the ends of them, where it was confined by two thumb-cleets; to this rope was attached, at any height that the depth of the ditch might require, a hook having four tails, and likewise hooks to them; these fixed into the four arms of the box, through which slid the poles for carrying it. The ends of the two leaping-poles were merely placed in the centre of the ditch, forming a pair of sheers, the apex of which was inclined on one side or the other by guys. The instrument was in this manner taken from one bank and landed on the other with the greatest gentleness; it was quickly unhooked, and men and apparatus passed over in the same manner. In this mode 263 ditches were passed between Sheerness and Greenwich.

On my arrival at the marshes, I found, from the nature of the soil, the greatest difficulty in adjusting the instruments; the movement of any of the men at a considerable distance caused a motion in the bubble, and the least alteration in my position whilst standing at the instrument shifted the bubble. In order to avoid moving, which heretofore was necessary to examine the level, I caused a square mirror, about eight inches by five, to be mounted on a mahogany frame, which permitted it to stand on the upper limb with the face placed at an angle of about 60° from the telescope, by which I could, without moving my head from the eye end of the telescope, read off the length of the bubble. This precaution was not however sufficient in some parts of the marshes, where the ground was of hardly more consistence than a bog, but invariably most unsubstantial where the most dry.

At these places, large pickets, from four to six feet in length, were driven in, having a groove on the iron head of each. On three of these, the iron-shod points of the stand rested; and after the observations were made, the pickets were drawn.

The pickets used for the stations were also of a larger class than those used in other parts; but the ground was occasionally so spongy, that it was with much difficulty that a picket could be driven, and frequently the iron heads of several would break off, before we could succeed in getting one down.

It will be seen by the Observation-book, that from a mean of pickets (page 5 to 11), the Queenborough and Sheerness marshes are in some parts 6.3652 below the northern standard; but these marshes are unlike the rest passed over, being particularly rugged and undulating.

In the commencement of the marshes between St. James and Yantlet Creek, it appears by a mean from picket 11 to 17, that the surface of the marshes is 5.8524 below the standard, or 1.7863 below mean spring tide high-water mark; and opposite Allhallows, by a mean from picket 35 to 44, the marshes are 3.7247 below the north standard, rising there $5.8524-3.7247=2.1277$.

About half a mile past the Decoy in St. Mary's marshes, nine miles distant from Sheerness, the marshes again fall: the mean from picket 75 to 86 gives 5.9916 below the north standard at Sheerness, and $5.9916-4.0661=1.9255$ below mean spring tide high-water mark.

Some distance past Cliff Canal, and between that and the Gravesend Canal, in Higham marshes, by a mean from 120 to 124, the surface is 6.6356 below the north standard, which -4.0661 gives 2.5695 below spring tide high-water mark, having fallen in a distance of seven miles and a half 2.9109, and in a distance of five miles 0.6440; the marshes between Northfleet and Greenhithe, by a mean from picket 208 to 211, are 7.4889 below the north standard, and $-4.0661=3.4228$ below spring tide high-water mark.

On the eastern side of Dartford Creek, the marshes, by a mean from picket 247 to 252, are 8.8676 below the north standard, therefore $8.8676-4.0661=4.8015$ below spring tide high-water mark.

On the western side of the marshes, as far as the mean from picket 256 to 259 will show, the marshes are 9.7207 below the north standard, and 5.6546 below spring tide high-water mark, and lower by 0.8531 than the marshes on the eastern side of Dartford Creek.

The marshes to the eastward, and in the immediate vicinity of the arsenal at Woolwich, are (from a mean of pickets 305 to 312) 10.1404 below north standard, and 6.0743 below spring tide high-water mark, and only 3.0107 above the mean level of the sea.

The only remaining pickets that were directly in the marshes, are from 347 to 352, the mean of which gives 9.6321 below north standard, and 5.5660 below spring tide high-water mark.

The following therefore is a statement of the depression of the marshes, from Sheerness towards the source of the Thames.

The Yantlet Creek marshes are 0.5128 higher than the Sheerness marshes.

The Allhallows marshes, in a distance of three miles, are 2.1277 higher than the Yantlet Creek, and 2.6405 higher than the Sheerness marshes.

The St. Mary's marshes are 0.1392 lower than the Yantlet Creek marshes, and 2.2669 below the Allhallows marshes, having fallen that quantity in a distance of three miles.

The Higham marshes are lower than those of Yantlet Creek by 0.7832, than those of Allhallows by 2.9109, and than those of St. Mary's by 0.6440, having fallen the last quantity in five miles.

The marshes between Northfleet and Greenhithe are lower than Yantlet Creek 1.6365, than Allhallows 3.7642, than St. Mary's 1.4973, than Higham 0.8533, being a fall of this last quantity in $6\frac{1}{2}$ miles.

On the eastern side of Dartford Creek, the marshes are 3.0152 below those of Yantlet Creek, 5.1429 of Allhallows, 2.8760 of St. Mary's, 2.2320 of Higham, and 1.3787 below the marshes between Northfleet and Gravesend; being a fall of the last quantity in $4\frac{1}{2}$ miles.

The marshes near Woolwich Arsenal to the eastward of the practising ground are 4.2880 below those of Yantlet Creek, 6.4157 of Allhallows, 4.1488 of St. Mary's, 3.5048 of Higham, 2.6515 of the marshes between Northfleet and Gravesend, and 1.2728 below the eastern Dartford Creek marshes, being a fall of 1.2728 in six miles.

The marshes at Greenwich, as far as the few observations I had the opportunity of making, are 0.5083 higher than those of Woolwich, therefore less that sum than the comparison of the Woolwich marshes.

At picket 131 (page 14), I intersected the Thames and Medway Canal, three miles from its mouth at the Thames.

The above picket was driven into the water's edge, another was at the same minute driven to the water's edge in the basin close to Gravesend. I then levelled along the banks, imagining that from a mean of simultaneous observa-

tions made at the two pickets at the water's edge, I should obtain the best proof to my levellings; the results (page 6 of Proof Levels) will show how little confidence is to be placed on water, as a true level, under such circumstances.

At the Lock Gate of the canal, close to the Thames (see page 15), I made some observations of the tides, and found, June 7th, that high-water mark at Gravesend was 1.1018 higher than at Sheerness, and June 8th, was 0.8367 higher; but these two observations are not to be depended on as giving a mean difference between the two places, as the height of the tides at Gravesend are much affected by any winds.

On the new pier at Gravesend, I caused one of the brass standards to be sunk in the granite on the eastern side, the face of which is 0.1828 below the north standard mark, and 3.8833 above mean spring tide high-water mark at Sheerness.

On my arrival at Greenwich Hospital, I commenced a set of branch levels, from thence to the Royal Observatory at Greenwich, in order to determine the height of that place above the level of the sea. From the abruptness of the ascent, the operation was very tedious; and I here found the advantage of the extra lens to the telescope, as there was seldom a distance of more than twelve or twenty feet between the pickets.

I levelled up to a small brass standard already placed for me by the direction of the Astronomer Royal in the block of stone immediately under the eye-end of the transit instrument pointing southward.

This standard (page 31 of Levels) is

140.6806 above the north standard at Sheerness.
153.8317 above the mean level of the sea.
144.7467 above the mean spring tide high-water mark.
162.3611 above the mean spring tide low-water mark.

These observations being completed, it occurred to the Astronomer Royal, after minutely examining my instrument, that it might be used as a proof in ascertaining the correctness of the horizontal point of the two mural circles.

By his directions I placed my level upon the high window-frame in front of and about eight feet from the object-end of the mural circle, pointed towards the north, and at such a height that (from a known principle in optics that all rays are parallel *,) I could intercept some of the rays from the object-glass.

* If any object be placed at the focus of a lens (viz. the wires), the emergent rays are parallel.

Having adjusted my level most exactly, I directed my telescope into the tube of the great telescope of the mural circle; and adjusting as for infinite distance, by placing a disc of white paper about an inch from the eye-end of the great telescope, I observed all the wires most distinctly. I then adjusted my horizontal wire for collimation.

The mean horizontal point was then taken, and the circle adjusted to that by the micrometer; and after again observing my instrument to be in perfect adjustment, I sought for the horizontal wire of the circle, and I was astonished and delighted to see so perfect a coincidence of the horizontal wires of the two instruments, that, until I slightly depressed the eye-end of my telescope, I could not see the horizontal wire of the circle separate from my own. The circle was then altered, and the wires were again made to coincide; the quantity was then read off, and found to agree within a very few hundredths of a second to the horizontal point.

The Astronomer Royal was present at these experiments, and expressed himself much pleased at the proof given of the coincidence of the two instruments.

From the stairs of Greenwich Hospital I crossed the river in order to level up to the different places where tide-registers had been kept.

After crossing the Isle of Dogs, I arrived at a spot on the south side of the lock of the City Canal at Limehouse Reach (see No. 381, page 26), which was 1.9008 above the north standard at Sheerness; and this spot was 3.8446 above the $\frac{\text{TRINITY HW}}{1800}$ marked on the face of the masonry. Therefore 3.8446 — 1.9008 = 1.9438 is the height of Trinity mark at the canal below northern standard at Sheerness, and 4.0661 (height of north standard above mean spring tide high-water mark) — 1.9438 = 2.1223, the height of Trinity mark above mean spring tide high-water mark at Sheerness.

This spot (picket No. 381) is also 0.5202 above a particularly high tide 21/ft. 11 in., 1827, marked on the masonry; but upon referring to the tide-register at Sheerness, of the 20th and 22nd of November 1827, no particular rise in the tide is to be remarked. It must therefore have been caused by land floods, which are the occasion of most of the extraordinary tides near London.

I next levelled to a standard placed at the West India Docks on the S.S.E. side, close to the entrance. This standard is 1.3018 above the northern standard, and 2.3367 above the index mark XXIII. Therefore 2.3367 − 1.3018 = 1.0349, the index mark below the north standard at Sheerness, and 4.0661 − 1.0349 = 3.0312 is the height of XXIII above mean spring tide high-water mark at Sheerness.

Not having succeeded in procuring a copy of the observations on the tides at the West India Docks, I cannot make any other comparison.

At the Regent's Canal Dock, in a very large stone near the office of the canal works (see page 26), I placed another standard mark. This standard is 2.4418 above the northern standard at Sheerness, and 2.2308 above the index mark XXI; therefore the index is 0.2110 above the north standard at Sheerness.

From two years' observations on the tides (1828—1829), the following are the results.

Mean Spring Tide High-Water Mark for

	1828.		1829.	
	Spring Tide, High Water.	High Water, Neap.	Spring Tide, High Water.	High Water, Neap.
	Ft. In.	Ft. In.	Ft. In.	Ft. In.
January	19.8[3]	15.	19.1	14.8
February	18.11	14.3	19.1[9]	13.11
March	19.9	14.3	19.5	14.8
April	19.9[8]	14.3	19.6	13.7
May	18.9	15.5	19.	14.9
June	18.10	15.7	18.9[5]	15.2
July	19.2[6]	15.9	18.9[6]	15.7
August	19.3	15.4[9]	19.3	14.7[6]
September	19.9	14.6	19.7	14.7
October	19.7	14.5		
November	19.1[5]	14.3	19.3	14.3
December	19.2[3]	13.9	18.7	14.9
Means	19.325	14.733	19.141	14.592

Spring 1829 ..... 19.141
—— 1828 ..... 19.325

Mean Spring Tide for two years 19.233

Neap 1829 ..... 14.592
—— 1828 ..... 14.733

Mean Neap Tide for two years 14.662

Mean Spring Tide 19.233
Mean Neap Tide 14.662

Mean High Water 16.947

The mean high-water mark, taking every high water through the months, is as follows:

	1828.	1829.
January	17.342	16.904
February	16.904	16.804
March	17.070	17.208
April	16.925	17.270
May	17.352	17.175
June	17.258	17.091
July	17.595	17.350
August	17.466	17.100
September	17.345	17.591
October	17.000	„
November	17.001	17.412
December	16.691	16.787
Means	17.1624	17.1538

17.1581

The difference therefore between the mean of spring and neap tide and the means of the months, as above, is (17.1581 − 16.947) 0.2111.

The index mark XXI at the canal is 0.2110 + 4.0661 = 4.2771 above spring tide high-water mark at Sheerness; and the spring tide high-water mark at the Regent's Canal being 19.233, XXI − 19.233 = 1.767, and 4.2771 − 1.767 = 2.5101, the height of mean spring tide high-water mark at the Regent's Canal above the same at Sheerness.

The index mark XXI is also 6.1321 above mean high-water mark at Sheerness.

And the mean high-water mark at Regent's Canal being

16.947, XXI − 16.947 = 4.053 and 6.1321 − 4.053 = 2.0791

or 17.1581, XXI − 17.1581 = 3.8419 and 6.1321 − 3.8419 = 2.2902 the height of mean high-water mark at the canal above the same at Sheerness.

And the mark XXI is 7.9761 above mean neap tide high-water mark at Sheerness. The mean neap tide mark at the canal being 14.662, XXI − 14.662 = 6.338 and 7.9761 − 6.338 = 1.6381, the height of mean neap tide high water at the canal above the same at Sheerness. Therefore the water of spring tides at the canal above spring tides at Sheerness is higher by 0.8720 than it is at neap tides at both places.

Having no observations of low-water mark at the canal, I have not the means of ascertaining the difference at the two places.

The next place where a comparison of the tides was made was at the London Docks, at No. 418 of the Levels (see page 27), being on an iron on the south-west pier of the main entrance close to the first lock. This iron was 3.7570 above the northern standard, and 5.7682 above the index mark by the side of the gates XXIIII, which mark answers to the 18-feet Trinity $\frac{\text{HW}}{1800}$

Therefore $3.7570 + 4.0661 - 5.7682 = 2.0549$ the height of XXIII above spring tide high-water mark at Sheerness. And $2.0549 + 1.855$ (the difference between spring tide and mean tide high-water mark at Sheerness) $= 3.9099$, the height of XXIII above mean high-water mark at Sheerness.

And $2.0549 + 3.699$ (difference between spring and neap tides at Sheerness), $= 5.7539$, the height of XXIII above neap tide at Sheerness.

And $2.0549 + 9.085$ (the difference between mean level and spring tide at Sheerness), $= 11.1399$, the height of XXIII above the mean level of the sea at Sheerness.

And $2.0549 + 17.615$ (difference between spring high and low water at Sheerness), $= 19.6699$, the height of XXIII above spring tide low-water mark at Sheerness.

By the kindness of Mr. LUBBOCK I have received the following results of twenty-six years' observations on the tides at the London Docks:

	Mean High Water.	Spring High Water.	Neap High Water.
	ft. in.	ft. in.	ft. in.
January	21.275	22.729	19.750
February	21.275	23.002	19.145
March	21.291	23.541	19.125
April	21.395	22.854	19.375
May	21.475	22.708	20.125
June	21.395	22.50	20.250
July	21.291	22.604	19.687
August	21.250	22.708	19.562
September	21.291	23.979	19.292
October	21.291	22.937	19.229
November	21.395	22.687	19.833
December	21.312	22.458	19.870
Mean	21.333	22.812	19.604

Therefore XXIII — 21.333 — 1.667.

And 3.9099 — 1.667 = 2.2429, the height of mean high-water mark at the London Docks above the same at Sheerness.

And XXIII — 22.812 = 0.188, therefore 2.0549 — 0.188 = 2.0361, the height of spring tide high-water mark at London Docks, above spring tide high-water mark at Sheerness.

And XXIII — 19.604 = 3.396, and 5.7539 — 3.396 = 2.3579, the height of neap tide high water at London Docks above the same at Sheerness.

No observations have been made on low-water mark; but from the Trinity mark it appears the spring tide low-water mark is considered to be 17.833 below Trinity mark, or rather below the high-water mark.

Therefore 22.812 — 17.833 = 4.979, and XXIII — 4.979 = 18.021, the height of XXIII above spring tide low-water mark.

And 19.6699 — 18.021 = 1.6679, the height of spring tide low water at London Docks, above spring tide low water at Sheerness.

Taking 22.812 and 4.979, the mean level of the sea is 13.896.

Therefore XXIII — 13.896 = 9.104, the height of XXIII above the mean level.

Then 11.1399 — 9.104 = 2.0359 gives the mean level at London Docks above the mean level of the sea.

The following is a Summary of the different heights.

Spring tide H. W. at London Docks, above the same at Sheerness				2.0361	0.2068
Mean H. W. mark	ditto	ditto	ditto	2.2429	
Neap tide Ditto	ditto	ditto	ditto	2.3579	0.1150
Spring tide low water	ditto	ditto	ditto	1.6679	0.6900
Mean level of the tides	ditto	ditto	ditto	2.0359	0.3680
Or taking more correctly the ½ difference between spring high and low water at Sheerness, the mean spring level is 10.8289 below XXXIII, therefore 10.8289 — 9.104 =				1.7249	

Note.—It seems by the above summary that as the water decreases in height, so the height of the water's surface at London Docks, above the same at Sheerness also decreases, with the exception of spring tide at London Docks and the neap tide; these are means, not of the highest tides, but of tides at a particular time of the moon's southing.

The next spot levelled to where any tidal observations were made, was St. Catherine's Docks, where a brass standard was placed close to the south-west side of the Dock-gates at the entrance.

This standard is 4.3143 above the north standard at Sheerness, and 6.2563 above the index mark XXVIII, upon a level with which is a —+— denoting Trinity $\frac{\text{HW}}{1800}$

Therefore 4.3143 + 4.0661 − 6.2563 = 2.1241, the height of this index or Trinity mark above mean spring tide high-water mark at Sheerness.

After passing along the Tower Wharf, and placing a standard mark in one of the large blocks of granite lately put down near the Traitor's Arch (see Levellings, page 28), I arrived (not without much vexatious interruption and annoyance in passing along Thames-street and Billingsgate) at the starlings of Old London Bridge, where I had some difficulty in making observations, owing to the tremor caused by the vehicles above. I levelled up to the Trinity mark on the western side of the bridge, and found as follows (see page 29):

No. 443 was 4.1047 below the north standard mark at Sheerness, and the Trinity $\frac{\text{HW}}{1800}$ was 1.9426 above No. 443.

Therefore 4.1047 − 1.9426 = 2.1621, the height of Trinity mark below north standard mark at Sheerness.

And 4.0661 − 2.1621 = 1.9040 gives the height of Trinity mark above the mean spring tide high-water mark at Sheerness.

Having now made all the observations that time and means afforded me, I concluded my levellings at a standard mark sunk in the large plinth of the landing-place (near the wall) of the stairs on the north-east side of the New London Bridge.

This standard was (page 29) 2.3967 below the north standard mark at Sheerness.

I have subjoined a list of the different Trinity marks I have observed, and their respective heights, and also of the brass standards placed by me, as well as various substantial points passed over in the levellings.

	Below North Stand. Sheerness.
Trinity high-water mark near the dock-gates of the City Canal .	1.9438
Ditto, answering to XXIII of the indices marked on the south-west side of the entrance to London Docks	2.0112
Ditto, at the west side of the lock-gate at the entrance to St. Catherine's Docks, answering to the index mark XXVIII . .	1.9420
Ditto, on the west side of London Bridge	2.1621

Finding so great a difference between the marks at the City Canal, London Docks, &c. and that at London Bridge, I levelled again in October last from St. Catherine's Docks to London Bridge, but found the same results. (See page 33.)

List of Standard Marks and other Points of Reference between Sheerness and London Bridge.

		Feet.
North standard at Sheerness		0.0000
South ditto ditto	higher	0.6048
Eastern ditto ditto	ditto	0.6735
Navy Well ditto ditto	lower	5.3921
Little ditto near tide-gauge at Sheerness	higher	0.4843
Queenborough standard	ditto	10.6029
St. James ditto	ditto	28.7454
The boundary post of Hoo, on a little eminence in the marshes, &c. (See page 10.)	below	2.6317
The 3 mile stone at the bank of canal. (See page 14)	above	2.4204
2 mile ditto	ditto	2.0038
1 mile ditto	ditto	1.1448
An iron clamp at second gate of Gravesend Canal. (See page 15)	ditto	0.5564
Brass standard mark on the pier at Gravesend . .	below	0.1828
The boundary stone of Swanscomb. (Page 18) . .	ditto	4.7578
Erith Church. (Page 20)	ditto	1.5669
Standard in Woolwich Arsenal	above	1.2019
A ⅂ in a corner-stone near the officers' guard-room	ditto	1.5576
The top of the 4½ mile post in the river (□) . . .	ditto	1.7182
□ on a stone at the west end of the dock-yard . .	ditto	1.9975

		Feet.
Brass standard in the dock-yard on the eastern point of mast slip	above	1.6545
A ⊼ in small north-east gateway of Greenwich College, from the main road	ditto	9.9975
Small brass standard underneath the transit at Greenwich Observatory	ditto	140.6897
Little brass standard on the plinth of the statue of George II. in Greenwich Hospital	ditto	5.1157
□ on one of the iron plates near the south side of the lock of the City Canal	ditto	1.9008
Brass standard at the West India Docks	ditto	1.3018
Brass standard at Regent's Canal	ditto	2.4418
A □ on the top of a granite post close outside the entrance from Ratcliffe Highway to the London Docks	ditto	16.0915
On the top of a gun in the Docks near the bridge of the eastern basin	ditto	13.1646
Brass standard St. Catherine's Docks	ditto	4.3143
Brass standard near the Traitor's Arch in the Tower	ditto	1.2854
Brass standard at landing-place of New London Bridge	below	2.3967

As it may be expected that I should state the manner in which I made the preceding observations, in order that a judgement may be formed of the confidence to be placed in them, I shall give a concise description of the field operations, and of the calculations necessary to complete the observations, and place them as they are in the book*.

After examining the ground, the particular line for carrying on the levellings was selected and marked out, when pickets were driven in at proper distances, according to the range of the instrument over the ground.

In the first part of the work the levelling was carried on as follows, the instrument being perfectly adjusted, and the station staves placed on their respective pickets. I made four observations, the telescope being in a different

* The field-book and all the other papers connected with the preceding operations are preserved at the Royal Society's Apartments.

position at each observation, and making a complete revolution in its collars: my assistant read off each, and noted it down in a book. I then went to the staff and examined its position on the picket, and read off the last observation myself; my assistant then read from his book the last observation; the two, of course, when correct, would correspond.

The telescope of the instrument was then reversed in its collars (which is a good check to the adjustment at each observation), and by the motion in azimuth directed to the next station staff, when the same mode was used in observing: the spot was marked, and the instrument moved to the next station, and the station staff turned half round in its collars, and gimbles ready for the next level. At the end of the day's work an additional picket was driven in about twelve or fifteen feet from that just used, and compared with it; this was to ensure the detection of any alteration in the pickets during the intervening time, either from mischief or accident, being compared before the commencement of the day's work. After a few days the ground was gone over again, generally from the opposite end, and two observations taken at each picket, which were sometimes more and sometimes less in number than in the former levellings; these were the proof-levels. The distances were then measured, and the necessary angles taken to lay down the work. In this manner proof levels were taken up to picket No. 111; but finding in this method that I was liable to great inconvenience and loss of time from many circumstances, and amongst others by the pickets being mischievously drawn or moved; and if a trivial mistake occurred in the levellings (of several days back perhaps), it was only detected at the summary of the levels. I determined to endeavour to adopt some method of proof, not liable to the inconveniences of the former.

I accordingly made use of the following mode, which I have found, after repeated trial, to be a most correct proof.

After having finished the four observations at one picket, I threw the instrument out of adjustment by the foot-screws; and after adjusting it again most correctly, I took a pair of observations; but instead of reading off the staff from the picket, it was now read from above, the staff being imaginarily continued to the length of ten feet. The mean of these observations, therefore, gave the complement to 10,000 of the true observation: by this mode much time was saved, and an error, if any, was immediately detected of a tenth of a

foot, or a foot (which are the errors most likely to occur), without the trouble of bringing the instrument and adjusting it two distinct times at the same spot. The mean of the four and the mean of the two observations ought of course to make 10,000 *. In the detail of the proof-levels, the correction for curvature is additive instead of subtractive; and in fact the whole operation is reversed, + standing for −.

In the course of my levellings, having instruments not generally used, I made some few notes, which I take leave here to transcribe.

In adjusting the station-staff, it is difficult to know when the zero on the vane is made to coincide with the horizontal wire of the telescope. I have found the most convenient and correct mode to be, to observe with the wire forming a diagonal to the lines on the vane, by which, when the staff is near, the two small black lines at each end of the vane could be seen; and when the vane was adjusted to the proper height, one of these lines was above and the other below the horizontal wire at equal distances, thus:

—At a greater distance, the two little white right-angled triangles, formed by the edges of the vane and bounded by the black semicircle, are very distinctly seen, the one above and the other below the (now-placed) horizontal wire, and can be compared in size with great nicety.

But in observing with the wire diagonally, great care must be taken that, by the vertical wire (the error of which, if any, will by practice be accurately known), the axis of the telescope shall bisect the centre of the staff.

In an instrument with a very sensitive level, there is usually some difficulty in adjusting the level. It arises from no fault in the level itself, but from a

* In the proof-levels it will be found, that generally, upon adding them to the mean of the four observations, there will be a quantity of from .0010 to .0060 to make up the 10,000. I could not discover the cause for some time.

The wires being at right angles to each other, of course have been at different distances from the eye-glass; but the difference not being much more than the thickness of a hair, I did not alter the eye-tube: however, upon examination, I found that at the usual distance I took levels, when I altered from extreme distinct vision of the horizontal wire to extreme distinct vision of the vertical wire (which was the one I used for proofs), it made a difference of from .0020 to .0050, the distinct vision of the vertical wire being that quantity lower than when observed with the same adjustment as for the horizontal wire.

difference in weight of those parts of the telescope outside the collar. I found this difficulty in my instrument: to remedy it, I measured off from one of the resting points half the distance between the two Ys on the collars, and suspended the telescope by a fine wire from this point, which was the proper centre of the telescope. I found the eye-end with the tube close in to be more than four ounces heavier than the object-end. To remedy this, I caused a thick ring of lead of the above weight to be placed inside the tube near the object-end, by which the telescope was balanced; and I found it, when adjusted, to reverse without differing a quarter of a degree.

Distinct vision is certainly desirable, but not so absolutely requisite as that there shall be no parallax of the wires. The best way to avoid this, is, after adjusting for distinct vision, to move the eye as far as the hole in the eye-piece will admit of, and observe if the wires have any motion over the object or vane: if so, it must be remedied by sliding the eye-tube in or out, until the objects appear motionless.

Mirage.

I have found that the tremulous motion or jumping in the air, termed as above, appears not to be caused so much by evaporation as probably by some oscillation in the particles of light: for I have remarked, when the sun shines brightly and is occasionally obscured by clouds, that while the sun is out, the tremor is so great as to prevent the possibility of making a correct observation; yet the moment the sun is obscured, the intermediate space between the instrument and object (provided the sun is obscured so as to cast a shadow the whole distance) will be immediately perfectly tranquil; and again, at the instant of the sun's appearance, the same tremor will be observed.

I have found this motion to be exactly equal above and below any object; for upon placing the wire of the telescope one half the distance between the extreme oscillations, whenever the sun is obscured, the wire will be found to bisect the object.

Description of the Observation-Book.

Each page of the book will be found to contain sixteen columns: the first and ninth contain the numbers of each staff or picket; the second and eleventh the mean from the rough book of the four observations at each staff; the third

and eleventh the distance in feet from staff to instrument and instrument to staff; the fourth and twelfth the correction for the curvature of the earth; the fifth and thirteenth the mean observations minus the curvature; the sixth and eighth are the length of the bubble and the thermometer attached to the instrument. This last is useful as a check to the bubble, which, when the instrument is moved suddenly, shakes into several small globules, that sometimes do not immediately unite again: this is detected by the length of the bubble, which ought to correspond to a certain degree of the thermometer. Column fourteen is the difference between columns five and thirteen; column fifteen is that difference + or −; and column sixteen is the amount of that difference added or subtracted, according to the sign, from the former quantity. These quantities are a continuation of heights above or below the first picket, or the northern standard in the dock-yard at Sheerness.

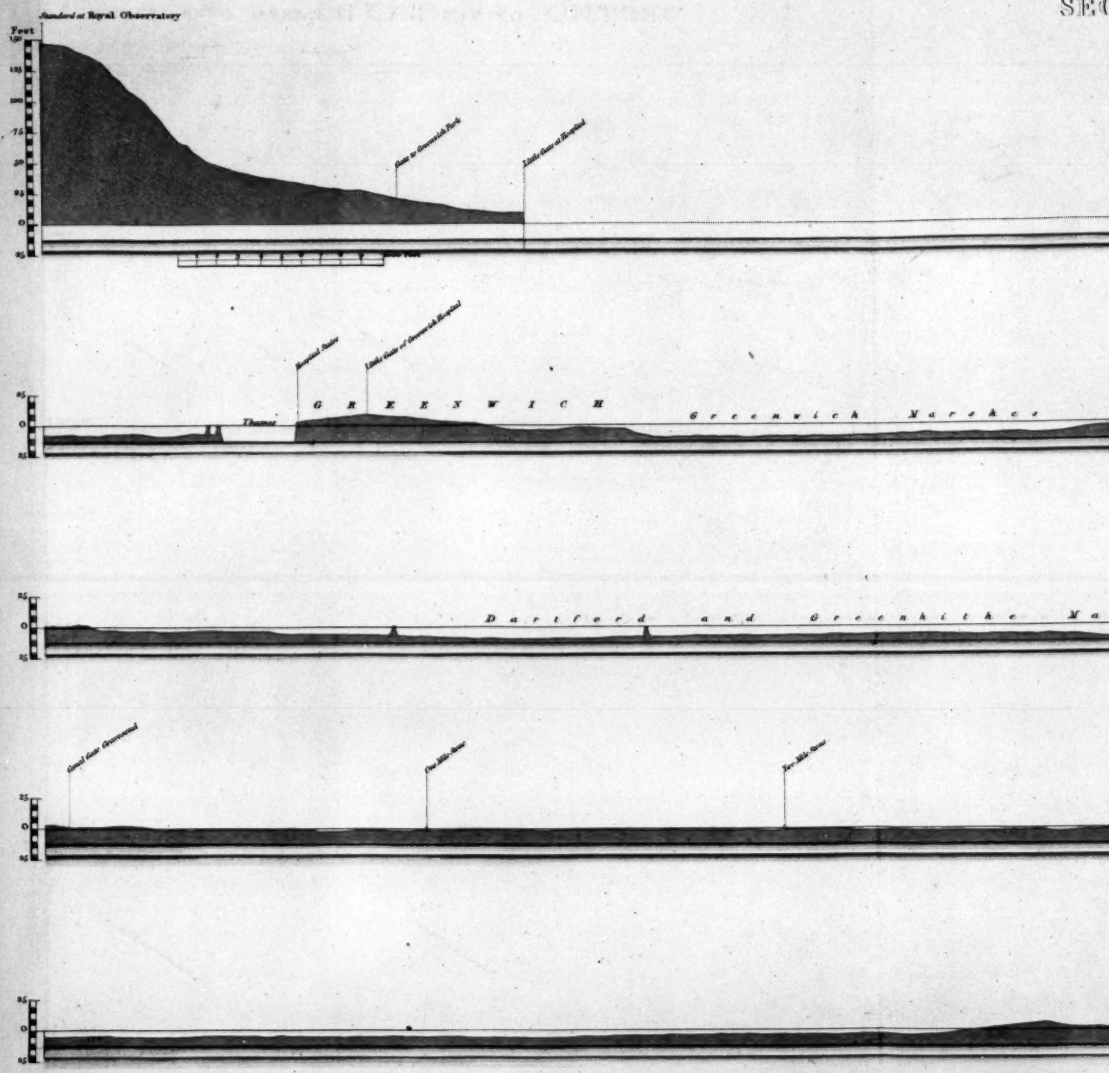
In order to prove the correctness of the different columns, they are summed up at the bottom, when the gross sum for curvature, being deducted from the gross sum of columns two and eleven, show the correctness of columns five and thirteen; and the difference between the sums of columns five and thirteen + or −, added to or subtracted from the little figures above the top line in column fifteen, ought to give the last true level in column sixteen at the bottom of the page.

The whole is further proved by taking the sums of columns five and thirteen of each page, and the difference of the whole amount of each gives a proof of the correctness of the whole work, by giving the difference at once between picket 1 and 445.

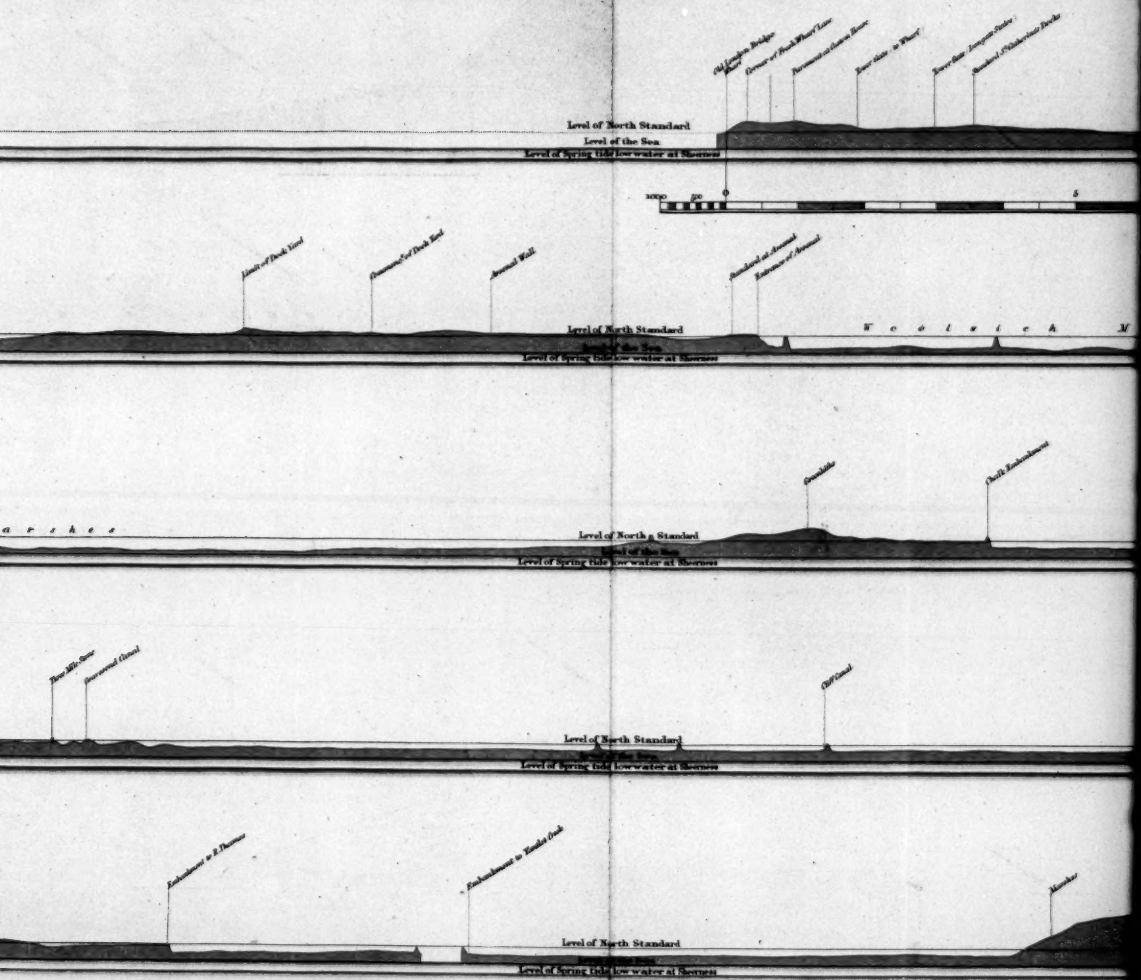
The same method is pursued in the proof-levels up to No. 112: after that, all the corrections become reversed, the curvature being additive (as the complement to 10,000 is read off on the staff). The difference between the proof-levels continue from standard at Sheerness to New London Bridge, and will be found to be 0.0110, and varying at different distances from Sheerness, but never more than 0.0300.

As there is not, in any work that I have seen, a correct table of curvatures of the earth for small distances, I have added one for every five feet from 60 to 1000.

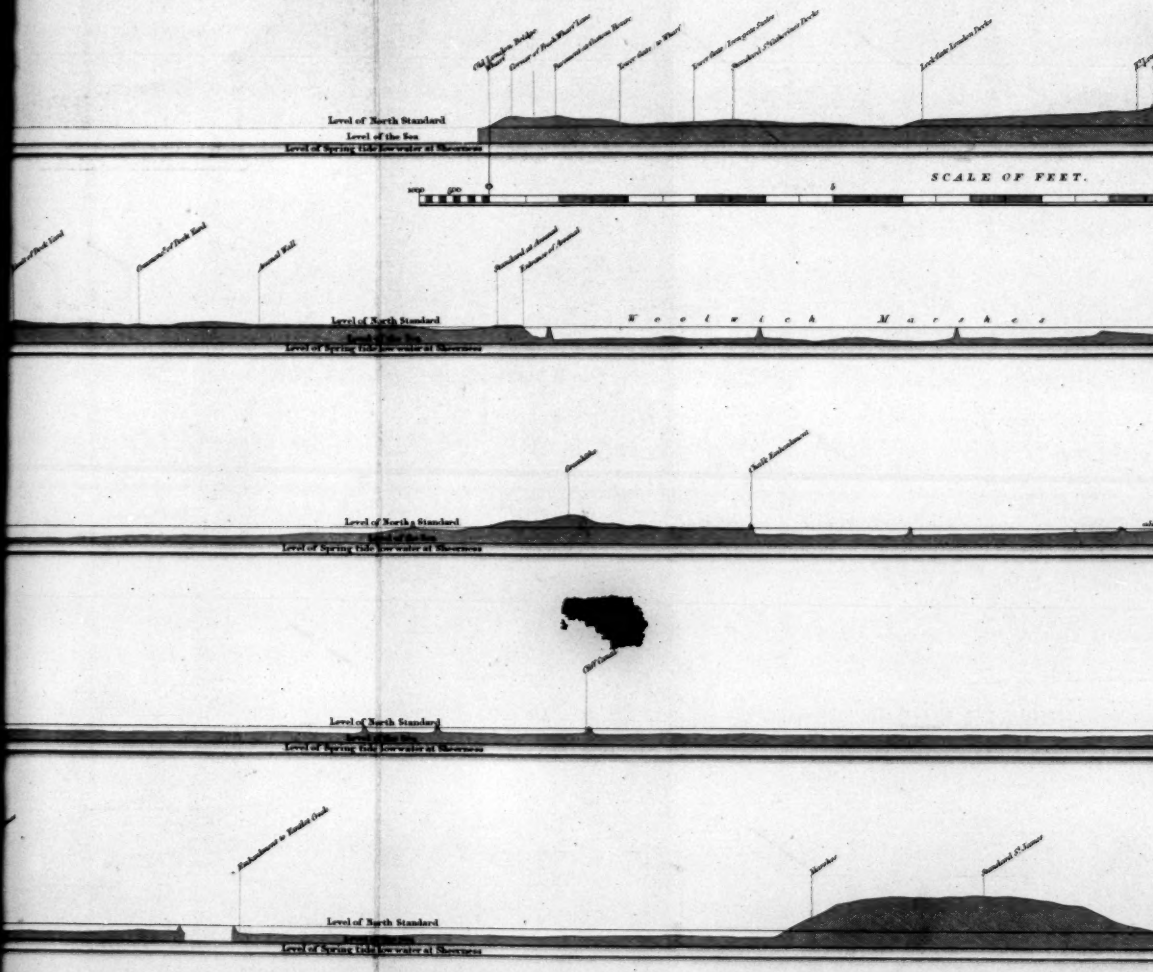
SEC


Standard at Royal Observatory
Feet
150
125
100
75
50
25
0
25
Gate to Greenwich Park
Little Gate at Hospital
Hospital Gates
Little Gate at Greenwich Hospital
G R E E N W I C H
Thames
Greenwich Marshes
Dartford and Greenhithe Ma
Canal Gate Greenwich
Two Mile Stone
Two Mile Stone

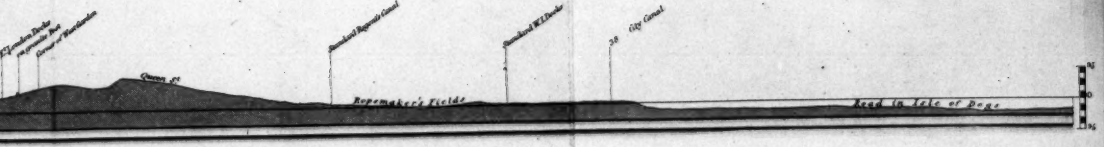
ECTION OF THE LAND PASSED OVER IN THE LEVELLINGS FROM SHEERNESS TO


Old Kingdom Bridge
Corner of Rook Wharf Lane
Entrance to Queen Street
Queen Street to Wharf
Queen Street Turnpike Gate
Standard of Watchorn Docks
Level of North Standard
Level of the Sea
Level of Spring tide low water at Sheerness
1000
500
0
5
Mount of Dock Yard
Pavement of Dock Yard
Arsenal Wall
Standard at Arsenal
Entrance of Arsenal
Level of North Standard
Level of the Sea
Level of Spring tide low water at Sheerness
W o o l w i c h A M
Granaries
Chalk Embankment
a r s h e s
Level of North Standard
Level of the Sea
Level of Spring tide low water at Sheerness
Three Mile Stone
Queenborough Channel
Old Canal
Level of North Standard
Level of the Sea
Level of Spring tide low water at Sheerness
Embankment at S. Thomas
Embankment to Rushes Creek
Marshes
Level of North Standard
Level of the Sea
Level of Spring tide low water at Sheerness

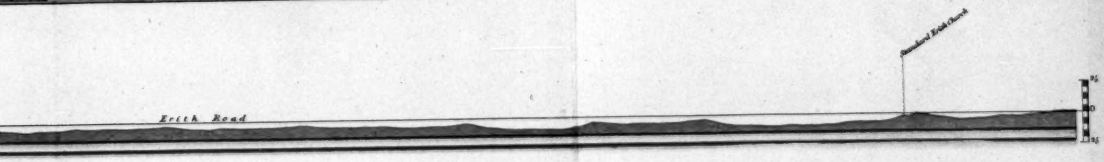
AND PASSED OVER IN THE LEVELLINGS FROM SHEERNESS TO LONDON BRIDG

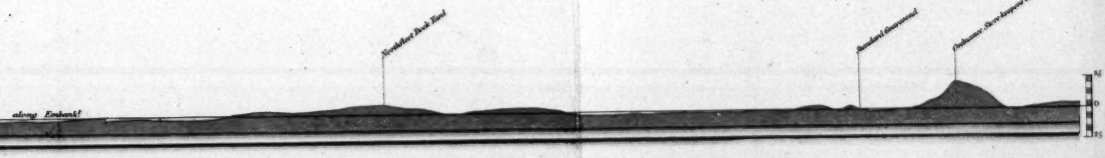

Level of North Standard
Level of the Sea
Level of Spring Tide low water at Sheerness
SCALE OF FEET.
Woolwich Marshes
Level of North Standard
Level of the Sea
Level of Spring Tide low water at Sheerness
Level of North Standard
Level of the Sea
Level of Spring Tide low water at Sheerness
Level of North Standard
Level of the Sea
Level of Spring Tide low water at Sheerness
Level of North Standard
Level of the Sea
Level of Spring Tide low water at Sheerness

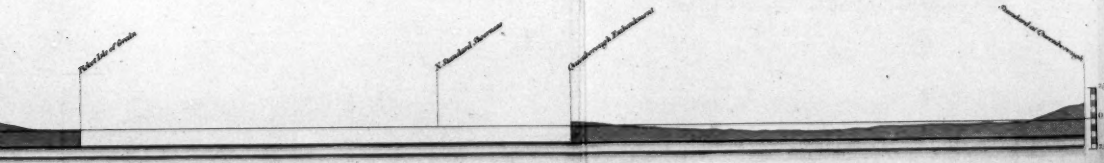
GE.



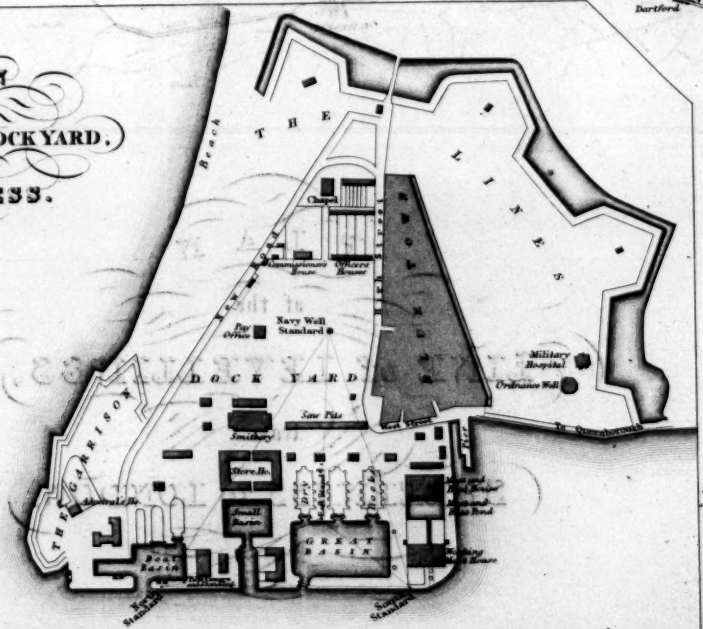
10 12 Thousand Feet.







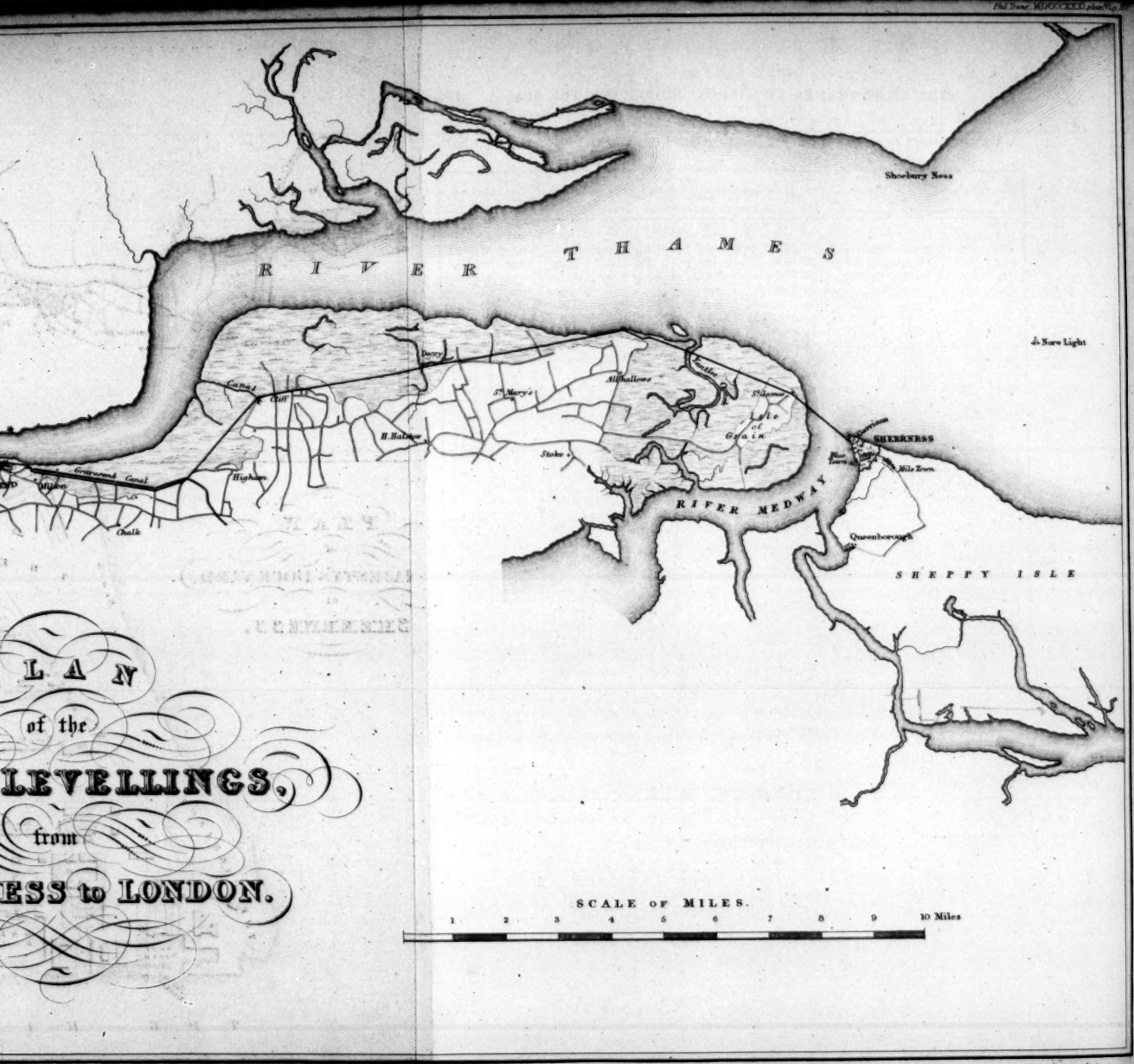
PLAN
of
HIS MAJESTY'S DOCK YARD,
at
SHEERNESS.



P

LINE of I

SHEERNE



LAN

of the

LEVELLINGS,

from

ESS to LONDON.

A Table of the Curvature of the Earth, the mean diameter being 41,807,803 feet.

Dist.	Curvature.	Dist.	Curvature.	Dist.	Curvature.	Dist.	Curvature.	Dist.	Curvature.	Dist.	Curvature.
Feet.		Feet.		Feet.		Feet.		Feet.		Feet.	
60	000086	220	001157	380	003453	540	006975	700	011721	860	017691
65	000101	225	001216	385	003544	545	007104	705	011888	865	017897
70	000117	230	001265	390	003637	550	007235	710	012057	870	018104
75	000134	235	001321	395	003730	555	007368	715	012228	875	018313
80	000153	240	001378	400	003827	560	007501	720	012399	880	018523
85	000173	245	001435	405	003923	565	007636	725	012572	885	018734
90	000194	250	001495	410	004021	570	007771	730	012746	890	018946
95	000216	255	001555	415	004119	575	007908	735	012921	895	019159
100	000239	260	001617	420	004219	580	008046	740	013098	900	019379
105	000264	265	001680	425	004321	585	008186	745	013275	905	019591
110	000290	270	001744	430	004422	590	008326	750	013454	910	019807
115	000316	275	001809	435	004526	595	008468	755	013634	915	020025
120	000344	280	001875	440	004621	600	008611	760	013816	920	020245
125	000374	285	001943	445	004735	605	008755	765	013998	925	020466
130	000404	290	002016	450	004843	610	008901	770	014181	930	020687
135	000436	295	002082	455	004952	615	009047	775	014366	935	020911
140	000469	300	002152	460	005060	620	009195	780	014552	940	021135
145	000503	305	002224	465	005171	625	009344	785	014740	945	021360
150	000538	310	002298	470	005284	630	009494	790	014928	950	021586
155	000575	315	002373	475	005397	635	009645	795	015118	955	021824
160	000612	320	002449	480	005511	640	009798	800	015308	960	022043
165	000651	325	002526	485	005626	645	009951	805	015500	965	022274
170	000691	330	002605	490	005743	650	010105	810	015693	970	022505
175	000732	335	002684	495	005861	655	010262	815	015888	975	022738
180	000775	340	002764	500	005978	660	010419	820	016083	980	022972
185	000818	345	002847	505	006100	665	010577	825	016280	985	023207
190	000863	350	002929	510	006221	670	010737	830	016478	990	023443
195	000909	355	003014	515	006324	675	010898	835	016677	995	023681
200	000957	360	003100	520	006467	680	011061	840	016877	1000	023919
205	001005	365	003187	525	006593	685	011223	845	017079		
210	001054	370	003275	530	006719	690	011383	850	017281		
215	001105	375	003363	535	006846	695	011554	855	017485		

NOTE.—I received the following rule for curvature and refraction together, from DAVIES GILBERT, Esq., and it will be found useful in ascertaining heights approximately.

Rule.—Assume the diameter of the earth as 10,000 instead of 7918 miles, and which will give the refraction about one-tenth the intercepted arc.

XI. *On the Variable Intensity of Terrestrial Magnetism, and the Influence of the Aurora Borealis upon it.* By Robert Were Fox. *Communicated by* Davies Gilbert, *Esq. M.P. V.P.R.S.*

Read March 17, 1831.

IN the annexed Table are given the results of a series of observations on the vibrations of the magnetic needle, which I undertook last summer, for the purpose of ascertaining whether its intensity is or is not affected by the changes in the earth's distance from the sun, or by its declination with respect to the plane of his equator; for, if we refer the nodes of the planetary orbits to this plane, there appears to be so considerable a degree of coincidence in most of them, as would seem to imply the existence of a more definite law than we are accustomed to attach to the abstract principle of gravitation*.

I am not at present prepared to say much respecting this part of my investigation; but I have obtained results, which appear to be interesting, relative to the variable force of the magnetic attraction, and the action of the aurora borealis on the direction and intensity of the needle.

I have used two needles, one possessing the north, and the other the south polarity in excess. To effect this, I have employed, in each case, a slip of seasoned oak, split with the grain, and suspended near the centre by unspun silk several inches long. This slip of wood serves as a support for a magnetic bar or needle, which is firmly riveted to it near one of its poles—say, for example, the north pole; whilst the south pole is at liberty to yield longitudinally to any

* I had been making numerous experiments with the same object in view some time before I commenced the series now given; but the results were unsatisfactory, from my not having employed a *stationary* magnetic apparatus, which should always be done in cases in which great accuracy is required. If the needle has been touched with the hand for one or two seconds only, its action is disturbed for some little time; indeed, it requires not a little practice to appreciate all the precautions that are necessary in vibrating the magnetic needle.

contracting or expanding force*. The magnetic bar is eight inches long, half an inch wide, and one tenth thick: its south pole extends rather more than two inches beyond the centre of suspension of the wooden support, so that there is nearly an equal portion of the southern half of the needle on each side of the centre, the north end being extended as a lever to control, in great measure, the magnetic movements of the needle†. A counterpoise of lead is placed at the other extremity of the support, sufficient to allow the north pole to dip at an angle of about forty degrees from the horizontal direction. The foregoing description also applies to the needle in which the south polarity predominates, except that the poles are reversed, the counterpoise being at the depressed end of the support. The former I shall distinguish as No. 1, and the latter as No. 2. Each is inclosed in a box of slate; as, if of metal, the action of the needles might have been disturbed: and the boxes are mounted steadily on bricks, in a room appropriated to them. A slip of glass at one end of each box enables me to observe the vibrations, which are from east to west, like the horizontal needle.

I have observed that the magnetic intensity is subject to frequent slight variations; but these I have been mostly unable to refer to any obvious cause, except when accompanied by the appearance of the aurora borealis, which evidently affected the needles on many occasions. Their vibrations, I think, generally became less rapid with a moist atmosphere, and more so when it was very dry; but I do not speak with full confidence on this point, as I have only recently made any regular hygrometrical observations. I might, perhaps, also mention changes of the wind, and snow storms, as being sometimes attended with fluctuations in the intensity of the needles. If the times of their vibration at different periods are compared, differences in their relative intensity will be noticed; that of No. 2. having on the average been diminishing, during the last three months nearly, in a more rapid ratio than No. 1. This must, I think, be considered an interesting fact, if confirmed by future observations. It does

* If the rivets which attach the magnetic bar to the support are placed at a proper distance from the acting pole, the equipoise of the two extremities will not be affected by the contraction or expansion of the bar.

† The neutralization of the pole would have been more fully effected had a smaller proportion of it extended beyond the centre of suspension; 1¼ to 1½ an inch would probably have been sufficient.

not seem easy to account for the discrepancies which not unfrequently occurred in the indications afforded by the two needles at the same time, and I hesitate to adopt any distinct conclusions from this circumstance without further experiments, which I intend to make with horizontal needles on the same principle.

I have endeavoured by various experiments to ascertain whether there is any decided and permanent difference in the directive force of the opposite poles;—for example, I have alternately neutralized in an equal degree the poles of a needle by means of a sliding axis, and suspended it horizontally by unspun silk, but the two ends vibrated in nearly equal times. Other needles on the same principle gave corresponding results, or at least the differences were so inconsiderable, and were so nearly compensated on an average of many observations at different times, that I think it may be safely assumed that there is an equality in the yearly mean of the magnetic intensity of the two poles separately considered. But on the hypothesis of a central magnetic force, ought not the north pole in this latitude to be acted on with much greater energy than the south? for if my experiments may be depended upon, the alternative can scarcely be adopted, of supposing that one pole of a needle is necessarily repelled as strongly as the other is attracted, since it appears that their relative intensity is not always the same. It therefore seems most reasonable to refer the phenomena of the earth's magnetism to the agency of electrical currents existing under its surface, as well as above it: indeed, I think it is impossible to doubt that the changes in the intensity and direction of the needle, which are often so transitory, must be due to meteorological causes *.

The aurora borealis which has frequently appeared more or less distinctly this winter, generally affected both needles during some part, and only a part, of the time of its being visible. On the seventh of last month, the aurora was seen from this place † as soon as it became dusk, and was still visible some time after midnight. It extended from N.N.E. to N.W. or W.N.W., and at intervals

* Is it not probable that the small anomalies which have been sometimes observed in the oscillations of the pendulum may be owing to the same causes which produce the much more considerable irregularities in the magnetic needle?

† Falmouth.

sent up streams of red and white light which occasionally nearly reached the zenith. These were most striking early in the evening, and more especially about eleven, or a little later, when the coruscations were beautiful. At 7 P.M.* I found the needles at 0, but soon after their north ends moved towards the east, and at 8 to 8½ their easterly variation was 1° 15′. They began to return westward at 8¾, and soon after 10 P.M. were again stationary in the magnetic meridian. At 11 to 11½ P.M., I found their intensity had diminished. Many instances are given in the table of the appearance of the aurora in the horizon about the magnetic north, and extending more or less considerably on each side of it. It was generally of a pale white, and sometimes I could perceive faint streams shooting upwards from the horizon a very few degrees, but more often I could not. It will be seen that considerable variations of the needle usually occurred in the course of the same evenings; and sometimes these variations took place on evenings when I did not remark any luminous appearance in the northern horizon.

All the variations at night were towards the east†, whether the aurora was actually visible or not; and hence may we not conclude, taking it for granted that it is an electrical phenomenon, and that it usually moves from about the north towards the south, that it must be of the nature of positive electricity? And, by a parity of reasoning, may we not assume the existence of an opposite state of electrical action by day, in order to account for the diurnal westerly variation, which is most considerable in the summer in these latitudes, when the aurora prevails about the south pole? This idea seems to be strengthened by a fact I have noticed, that the magnetic intensity is usually less considerable in summer before the middle of the day than it is afterwards; so that the minimum intensity commonly occurs some hours before the maximum temperature.

It is evident that the elevation of the aurora must often be exceedingly great, probably much more than a thousand miles, as it seems to be generally seen from places very distant from each other at the same time, and in nearly the same

* I did not observe the needles before 7 P.M.

† I have since observed a slight westerly variation in the needle at night, but this seems to be of rare occurrence.

direction: thus, for example, the observations made on the beautiful aurora of the 7th ultimo from the vicinity of London, and of this place, seem very nearly to accord in almost every respect.

In conclusion, I will venture to express a hope that the subject of terrestrial magnetism may obtain the attention which it appears to deserve, and that experiments may be made at the same time in different countries, in order to develop its more obscure properties. It will then perhaps be found that its relations to other natural phenomena are as extensive as they are interesting and important. At any rate, it seems probable that some light might be thrown on the hypothesis of electrical currents under and above the surface of the earth, and their relative influence on the magnetic needle, if observations on its intensity were to be made on small islands, as remote as possible from any large tracts of land, and the results compared with others obtained by the same apparatus, on extensive continents at stations as nearly as may be in the same magnetic parallels: or, instead of employing the same apparatus, several magnetic needles might be forwarded to different places, after having been carefully compared with a standard needle; and with these, simultaneous observations might be made, not only in the same parallels, but likewise in different parallels of latitude and longitude in both hemispheres.

A Table showing the Times of Vibration of two Magnetic Needles, described in the annexed paper, in the neighbourhood of Falmouth.

Needle No. 1.					Needle No. 2.				Baro-meter.	Ther. meter in room with needles.	Ther. meter out, ex-posed to the north.	Lunar phases.	Extraordinary variations of the needles by night.	Winds.	General Remarks.
Time of Observation.			Arcs 80° to begin, ended as under.	Number of seconds in making 80 vibrations.	Time of Obs.		Arcs 80° to begin, ended as under.	Number of seconds in making 80 vibrations.							
1830.	A.M.	P.M.			A.M.	P.M.									
Aug. 2	.....	10	.....	1183+.00											
3	10	.....	39.00	1179+.00											
4	.....	10	39.00	1184+.00	.....	.....	.....	.....	.....	.....	.....	○	....	.....	Clear.
5	10	.....	39.00	1186+.00											
6	.....	10	39.00	1181+.00	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	Fine weather.
9	.....	10	39.00	1183+.00											
Mean for the month			.....	1182.66											
[a]Oct. 7	.....	10	38.30	1180+.00	.....	10.30	43.00	1137.00	30.30	60.00	.....	☾ 8th	.....	.....	[a] Suspended nearly two months in consequence of absence from home.
9	.....	3.30	38.30	1180+.00	.....	4	43.00	1132.00	30.40	.....	.....	.....	.....	.....	Fine clear weather.
13	.....	10	39.00	1177+.00	.....	10.30	43.00	1131.00							
15	.....	10	39.00	1180+.00	.....	11	43.00	1133.00							
Mean for the month			.....	1179.25	.....	.....	.....	1133.25	.....	.....	.....	● 16th			
[b]Nov. 8	.....	8	39.00	1177+.00	.....	8.30	42.00	1127.50	.....	.....	.....	.....	.....	.....	[b] Again suspended from absence.
12	.....	8	39.00	1178.00	.....	8.15	43.00	1130.00							
13	10.30	.....	39.00	1180.00	11	.....	44.00	1134.50	.....	.....	.....	.....	.....	S	A great storm with rain.
15	.....	7.30	39.00	1179.00	.....	8.30	43.00	1134.00	29.20	58.00	.....	●	.....	SW	A great storm without rain.
16	10	.....	40.00	1182.00	10.30	.....	44.00	1136.00	28.50	58.00	.....	.....	.....	„	Calm throughout the day, but a great storm at night.
19	.....	7.30	39.00	1176.00	.....	9.45	44.00	1133.00	30.00	54.00	.....	.....	.....	„	Fine.
22	.....	8	39.00	1176.00	.....	9	44.00	1131.00	29.90	55.00	.....	.....	.....	„	Nearly calm and clear weather.
23	10	.....	39.30	1176.00	10.30	.....	44.00	1130.00	30.00	54.00	.....	☽	.....	„	Nearly calm and clear weather.
24	.....	2.15	40.00	1177.00	.....	2.30	44.00	1126.50	30.20	55.00	.....	.....	.....	SE	A strong dry wind.
25	11.30	.....	40.00	1177.00	10	.....	44.30	1125.00	30.10	55.00	.....	.....	.....	„	A strong dry wind, very clear.
26	11.30	.....	40.30	1175.00	noon		44.30	1123.00	29.80	52.00	.....	.....	.....	„	Clouded, and wind stronger—some rain.
27	.....	11	40.00	1176.00	.....	11.30	44.30	1124.00	28.90	52.00	.....	.....	.....	ESE	Rain in morning, evening clear.
29	10.30	.....	39.30	1180.00	11	.....	44.00	1128.00	29.40	56.00	.....	.....	.....	.....	Strong wind and dry weather.
Mean for the month			.....	1177.61	.....	.....	.....	1129.38	29.60	54.90		○ 30th			

Dec. 1	.....	10	39.30	1178.00	.....	10.30	45.00	1127.00	29.60	54.00	.....	.....	.....	SE	Wind strong—clouded.
2	10	.. ...	39.30	1178.00	10.20	.....	45.00	1127.00	29.30	54.00	.....	.....	.....	ESE	Wind not strong—rain.
3	.....	10	40.00	1177.00	.....	10.30	44.30	1126.00	29.30	54.00	.....	.....	.....	NE	Clouded, and light wind.
4	10.30	.....	39.30	1176.00	10.50	.....	45.30	1126.00	29.50	54.00	.....	.....	.....	SE	Clouded, and light wind.
5	.....	9.30	40.00	1176.00	.....	10	45.00	1128.00	28.10	54.00	.....	.....	.....	„	A great storm to-day, which abated in the evening.
7	9.30	.....	41.00	1175.50	10	.....	46.00	1128.00	28.40	55.00	.....	☾	.....	NE	Rain.
8	.....	8.30	41.00	1174.00	.....	7.30	45.30	1127.00	28.70	54.00	.....	.....	.....	SE	Rain.
9	.....	10	41.30	1173.00	.....	10.20	45.30	1125.00	28.60	52.00	.....	.....	.....	NE	Clouded.
10	9	.....	41.30	1173.00	.....	.....	46.00	1125.50	28.80	51.00	.....	.....	.....	NW	Showers.
11	.....	7.30	41.00	1175.75	.....	7	46.00	1126.50	29.20	49.50	.....	.....	.....	W	Strong squalls with hail showers.
„	.....	11.15	42.00	*1177.30	.....	11.40	45.30	*1129.00	*29.*20	*49.00	.....	.....	.....	„	Aurora visible this evening; the variation not observed.
12	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	6 P.M. 45′ E.	NNE	Aurora again visible in N.N.E. to N.W. from 6 to 11 P.M.
13	.....	4	41.00	1172.00	.....	4.30	46.00	1122.50	30.30	47.50	.....	.....	.....	SW	Hard frost in the morning, but now gone.
14	.....	11	41.30	1172.00	.....	.....	46.00	1123.50	30.50	48.00	.....	.....	.....	SSW	Sunshine.
15	.....	10	40.00	1172.00	.....	.....	45.00	1124.00	30.45	49.00	.....	●	.....	S	
16	9	.....	41.00	1174.00	9.30	.....	44.30	1126.00	30.25	50.00	.....	.....	.....	N	Clouded, and small rain.
„	.....	9.30	40.45	*1171.00	.....	10	44.30	*1124.00	„	„	.....	.....	9 P.M. 30′ E. 10½ P.M. 15′ E.	ESE	Strong wind, and clear.
17	9	.....	41.15	1172.00	9.30	.....	44.30	1124.50	29.30	49.00	.....	.....	.....	NNE	Nearly calm.
18	.....	10.30	41.15	1169.50	.....	11.30	45.00	1122.50	30.30	46.00	.....	.....	.....	W	
20	9	.....	41.30	1173.50	9.30	.....	45.15	1126.00	29.50	47.50	.....	.....	.....	N	Showers.
21	.....	10	41.15	1174.00	.....	10.45	45.15	1126.50	29.30	49.00	.....	.....	10 P.M. 20′ E.	NNW	Clouded.
22	.....	7	42.00	1173.50	.....	7.30	45.30	1126.00	29.40	49.00	.....	☽	.....	WNW	Strong gusts of wind, and rain.
23	9	.....	41.30	1171.50	9.30	.....	46.00	1125.00	29.45	46.50	.....	.....	.....	NW	Hail showers.
„	.....	11	40.30	*1168.50	.....	11.30	45.30	*1123.00	*29.50	46.00	30.00	.....	9½ P.M. 30′ E.	N	Snow showers.
24	.....	11	40.30	1167.00	.....	11.30	45.30	1128.00	29.50	41.00	27.00	.....	.....	„	Mostly clouded; deep snow on the ground.
25	.....	2	40.15	1167.00	.....	3.15	45.30	1128.50	29.30	41.00	25.00	.....	.....	„	Clear weather, and deep snow on the ground.
„	.....	9.30	40.00	*1173.50	.....	10	46.00	*1134.00	*29.20	*41.00	*33.50	.....	94 P.M. 35′ E. 10 P.M. 50′ E. 12 P.M. 30′ E.	„	Snow and sleet falling.
27	9.15	.....	40.30	1171.50	10	.....	46.30	1131.00	28.70	41.50	36.00	.....	.....	W	Clouded.
„	.....	11	40.00	*1172.00	.....	11.30	46.00	*1132.00	*28.70	*42.00	*39.00	... ..	9 P.M. 15′ E.	SW	Clouded, and rain occasionally.
28	9	.....	40.30	1171.00	10	.....	45.30	1132.00	28.90	42.50	39.00	.....	.....	W	Clouded, and showers of hail and rain.
29	9	.....	40.15	1172.00	9.30	.....	44.45	1132.00	29.50	43.00	39.00	○	.....	SW	Rather hazy.
„	.....	10.45	40.40	*1174.00	.....	10	45.00	*1133.00	*29.35	*44.00	*44.00	.....	9 P.M. 25′ E. 94 P.M. 50′ E. 104 P.M. 15′ E.	S to E	Just commencing to blow strongly—showers.
30	9.30	.....	40.30	1174.50	10	.....	44.30	1134.50	28.85	44.00	47.00	.....	.....	E	Strong wind and much rain.
31	9	.....	41.30	1176.00	10.30	.....	45.00	1134.00	29.00	46.50	45.00	.....	.....	W	Clouded and damp.
„	.....	11.30	41.00	*1173.00	11	.....	45.30	*1131.00	*29.50	*45.50	*38.50	.....	.....	W	Clear, and fine night.
Mean for the month		.....		1173.80	.....	.....	.....	1126.72	29.39	48.61	36.60				

Table of the Vibrations, &c. (Continued.)

Needle No. 1.					Needle No. 2.				Barometer.	Thermometer in room with needles.	Thermometer out exposed to the north.	Lunar phases.	Extraordinary variations of the needles by night.	Winds.	General Remarks.
Time of Obs.			Arcs 80° to begin, ended as under.	Number of seconds in making 80 vibrations.	Time of Obs.		Arcs 80° to begin, ended as under.	Number of seconds in making 80 vibrations.							
1831.	A.M.	P.M.			A.M.	P.M.									
Jan. 1	.....	9	41.00	1174.00	9.30	.....	44.30	1133.50	29.40	48.00	50.00	.....	.....	S	A strong wind and some rain.
3	.....	10	41.45	1175.00	.....	11.30	45.00	1134.50	29.50	51.50	50.00	.....	$6\frac{1}{4}$ P.M. 15′ E.	E	Much wind—clouded.
4	9	.....	41.00	1175.00	9.30	.....	45.00	1133.25	29.50	50.00	49.00	.....	.....	„	Clouded.
5	.....	4	41.30	1175.50	.....	3.30	45.00	1134.25	29.65	51.50	51.00	☾			
6	.....	10	40.30	1173.50	.....	10.30	45.40	1133.00	30.50	49.00	41.00	.....	.....	NE	Clear weather.
7	9	.....	40.30	1171.75	9.15	.....	45.00	1131.50	30.60	48.00	41.00	.....	.....	E	Clear weather.
„	.....	11.30	41.00	*1177.30	.....	11	45.00	*1135.00	*30.60	*48.00	*39.00	.....	7 P.M. 0. 8 to 8½ P.M. 1.15′ E. 8½ P.M. 1.10′ E. 9 P.M. 40″ E. 9.30 P.M. 30″ E. 10 P.M. 5′ E.	„	Clear weather—Aurora visible at 5½ P.M. in N. & N.W., and afterwards extended to N.E. & W.N.W.; it was seen after midnight.
8	9	.....	40.30	1173.50	9.30	.....	44.30	1132.25	30.65	46.00	42.00	.....	9½ P.M. 15′ E. 11 P.M. 0′ E.	„	Hoar frost in the morning, and afterwards clear—a faint white light in N. & N.W. most of the evening, till after midnight.
10	.....	11.30	41.30	1173.00	.....	12	45.00	1133.00	29.90	46.50	46.50	.....	10 P.M. 10′ E. 10½ P.M. 20′ E. 11 P.M. 10′ E.	NE	Luminous in N.N.W. this evening.
11	.....	.....	40.30	1172.50	9.30	.....	45.00	1133.00	29.92	46.00	39.00	.....	.....	„	Clouded.
„	.....	10	41.00	*1172.00	.....	11	45.30	*1126.75	*29.94	*45.00	*38.00	.....	No variation observed.	„	Clear—faintly luminous in magnetic meridian.
„	.....	11.30	41.30	*1172.50	.....	11.45	45.00	*1124.50	*29.94	*45.00	*38.00				
12	.....	10.30	41.30	1172.00	.....	11	45.00	1133.00	29.90	45.00	45.50	.....	.....	ESE	Clouded.
13	9	.....	41.30	1173.00	9.15	.....	44.30	1132.00	29.92	46.00	45.00	.....	.....	„	Clouded.
14	9	.....	41.00	1172.75	9.30	.....	45.00	1132.00	29.95	46.00	45.00	●	.....	SE	Clouded.
„	.....	11.30	41.30	*1171.50	.....	12	45.00	*1133.00	„	„	„	.....	8 P.M. 10′ E. 8½ P.M. 25′ E. 9 P.M. 10′ E.	„	Pale white Aurora this evening in N. & N.W.
15	noon		41.15	1172.00	.....	1	45.30	1131.50	29.70	46.50	44.00	.....	.....	„	Clouded.
17	9	.....	41.15	1174.50	9.30	.....	45.00	1133.75	29.20	48.00	51.00	.....	.....	SE	Clouded, and some rain.
18	9	.....	41.45	1175.50	9.30	.....	45.15	1134.00	29.00	50.00	52.00	.....	.....	SW	Clouds and mists, with some rain.
19	.....	2.30	41.30	1175.50	.....	4	45.00	1134.50	29.20	51.00	50.50	.....	.....	SSE	Watery clouds and damp.
20	.....	10.15	41.30	1176.00	.....	10.20	45.00	1135.00	28.40	51.50	50.00	.....	.....	SE	Rain.
21	9	.....	42.00	1176.00	9.45	.....	45.00	1134.50	28.35	52.00	50.00	☽	.....	„	Rain and strong wind.
„	.....	10.15	41.30	*1176.50	.....	10.45	45.00	*1134.50	*28.50	*53.00	*50.00	.....	10 P.M. 15′ E.	„	Nearly calm.
25	.....	7	41.00	1171.00	.....	7.30	45.00	1130.50	30.00	46.00	35.00	.....	.....	NNW	Some snow fell in the night.
26	.....	5.45	40.30	1168.50	.....	6.15	45.00	1129.50	30.20	45.00	35.00	.....	.....	N	Fine weather, and a little snow on the ground.
28	.....	7.30	41.00	1170.00	.....	8	44.30	1129.00	29.63	45.00	39.00	○	.....	NW	Cloudy with showers.
29	.....	7.30	41.00	1169.00	.....	8	44.30	1130.00	29.63	44.00	39.00	.....	.....	SSE	Rain and sleet.
31	.....	8	41.00	1170.50	.....	8.30	45.00	1130.00	28.17	44.50	40.00	.....	.....	SW	Much rain to-day.
Mean for the month		.....		1173.04	.....	.....	.....	1132.54	29.60	47.70	45.24				

Feb. 1	.....	8	41.00	1169.50	.....	8.30	45.00	1129.00	28.90	42.00	36.00	.....	.....	NW	Squalls with sleet and hail showers.
2	.....	8	41.00	1169.50	.....	8.30	45.30	1129.00	28.90	42.00	36.00	.....	.....	"	Showers occasionally, but clear in the evening.
3	.....	9	41.00	1169.50	.....	10	45.30	1130.50	28.70	43.00	39.00	.....	.....	"	Rain.
4	.....	9.45	41.00	1170.00	.....	10.15	45.00	1130.50	29.55	43.00	39.00	☾	.....	WNW	Clear.
5	.....	7.30	41.00	1170.00	.....	7.45	45.00	1130.50	29.85	44.50	41.00	.....	.....	"	Clear.
7	.....	7	41.00	1177.00	.....	7.30	45.00	1136.50	29.43	50.00	52.00	.....	.....	SSW	Clouded, and a little rain.
8	9	.....	41.15	1177.50	9.30	.....	45.00	1137.00	29.50	55.00	55.50	.....	.....	"	Rain.
9	.....	9.15	40.30	1174.50	.....	9.45	44.30	1134.00	29.90	56.50	55.00	.....	.....	"	Clouded, and some rain.
10	8.15	.....	41.00	1174.50	8.30	.....	45.00	1135.50	29.90	55.00	57.00	.....	.....	SW	Fine weather.
"	.....	10	40.30	*1174.00	.....	9.30	44.30	*1135.00	*30.00	*55.00	*52.00	.....	Slight variation, E.	"	Faintly luminous in N.W.
11	10.15	.....	41.15	1175.50	10.30	.....	45.00	1136.50	30.15	55.00	52.00	.....	.....	WSW	Clouded and damp.
12	9	.....	41.15	1176.50	9.30	.....	45.00	1137.00	30.20	53.00	53.50	●	.....	"	Clouded and damp.
14	.....	6.45	41.00	1175.00	.....	7	45.00	1135.50	30.00	51.00	50.50	.....	.....	S	Clouded, and some rain.
15	9.15	.....	41.30	1175.50	10	.....	45.00	1136.00	29.85	50.50	50.00	.....	11 P.M. 25′ E. 11¼ P.M. 0′ E.	"	Clouded, and some rain—at 11 P.M. horizon in N.N.W. a little luminous, but much increased at 11¾ to 12.
16	8.15	.....	41.30	1176.00	8.30	.....	45.00	1136.00	29.80	51.00	48.00	.....	.....	NW	Fine morning, afterwards clouded.
17	8	.....	41.15	1173.75	8.30	.....	45.30	1134.50	30.10	48.50	48.00	.....	7 P.M. 30′ E. 7¼ P.M. 15′ E. 7¾ P.M. 0′ E.	"	Fine weather, slightly luminous in N.N.W. at 7 P.M.
18	8	.....	41.00	1174.00	8.20	.....	45.30	1133.00	30.25	48.00	45.00	.....	.....	W	Clear weather.
19	9.45	.....	41.15	1174.00	10.30	.....	45.30	1134.50	30.30	49.50	51.00	☽	.....	NW	Clouded.
21	.....	7.15	41.00	1173.00	.....	7.45	45.30	1133.20	30.05	47.00	47.00	.....	.....	WNW	Fine day—some clouds.
22	8	.....	41.15	1174.50	8.30	.....	45.45	1135.40	30.00	48.00	51.00	.....	.....	N	Some rain and damp.
23	8	.....	41.00	1174.50	8.20	.....	45.15	1134.60	30.37	47.00	44.00	.....	.....	NE	Clouded, but dry weather.
24	8	.....	40.30	1174.00	8.30	.....	45.30	1135.00	30.15	47.50	50.00	.....	.....	SW	Clouded—some showers.
25	8	.....	41.00	1174.00	8.20	.....	46.00	1135.50	29.87	48.00	47.00	.....	.....	NW	Clouded—damp.
26	.....	6	41.30	1175.00	.....	6.20	46.00	1136.50	29.35	49.00	48.00	○	.....	WSW	Much rain.
28	.....	9.30	41.00	1173.50	.....	10	46.00	1134.50	29.73	48.00	42.00	.....	.....	NNW	Clear this evening and dry.
Mean for the month				1173.75	.....	.....	.....	1134.17	29.78	48.83	47.37				

Note 1.—I have added 8″ to each of the numbers marked thus †, in consequence of a slight alteration having been made in the dip of needle No. 1, on the 8th of November, which appeared to add at least eight seconds to the time of its performing 80 vibrations.

2.—The numbers in the above Table marked thus *, are not comprised in the monthly averages, as I thought it best to include in the mean, only one observation for each day, and most of the results so marked are rather anomalous, or were obtained about the time when the needle varied considerably to the eastward. I may here remark that I had four needles, from which I noted the variations.

3.—There is a considerable difference in the arcs described by the two needles: this I attribute to the circumstance of No. 1 being not so heavy as No. 2, the latter having of course required a more considerable counterpoise, in order to give an elevation to its south pole equal to the depression of the north pole of No. 1.

No. 2 weighs, including the support and counterpoise ..... 898 grains.
No. 1 weighs, including the support and counterpoise ..... 854 ——

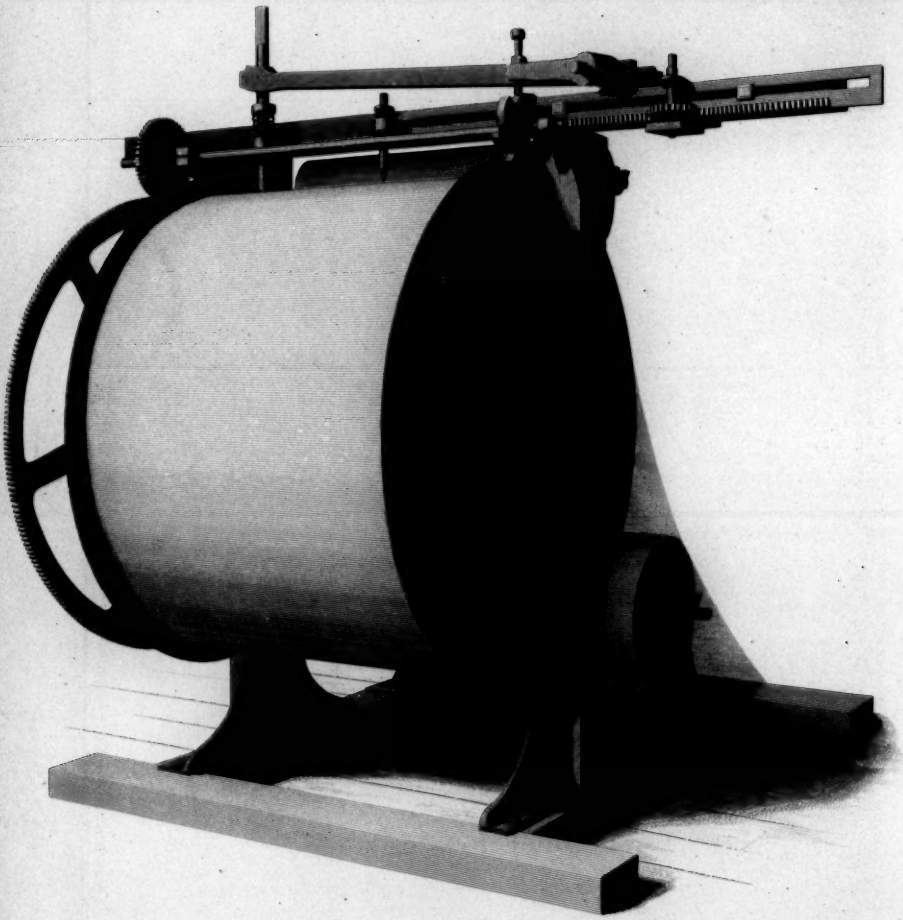
Difference ..... 44 grains.

On adding 202 grains to the weight of No 1, and retaining it at the same dip, instead of 41° 15′, it described an arc of 50°, after having performed eighty vibrations. The difference in the times of the two needles, I attribute to No. 2 being more strongly magnetic than No. 1.

PLATE 4.

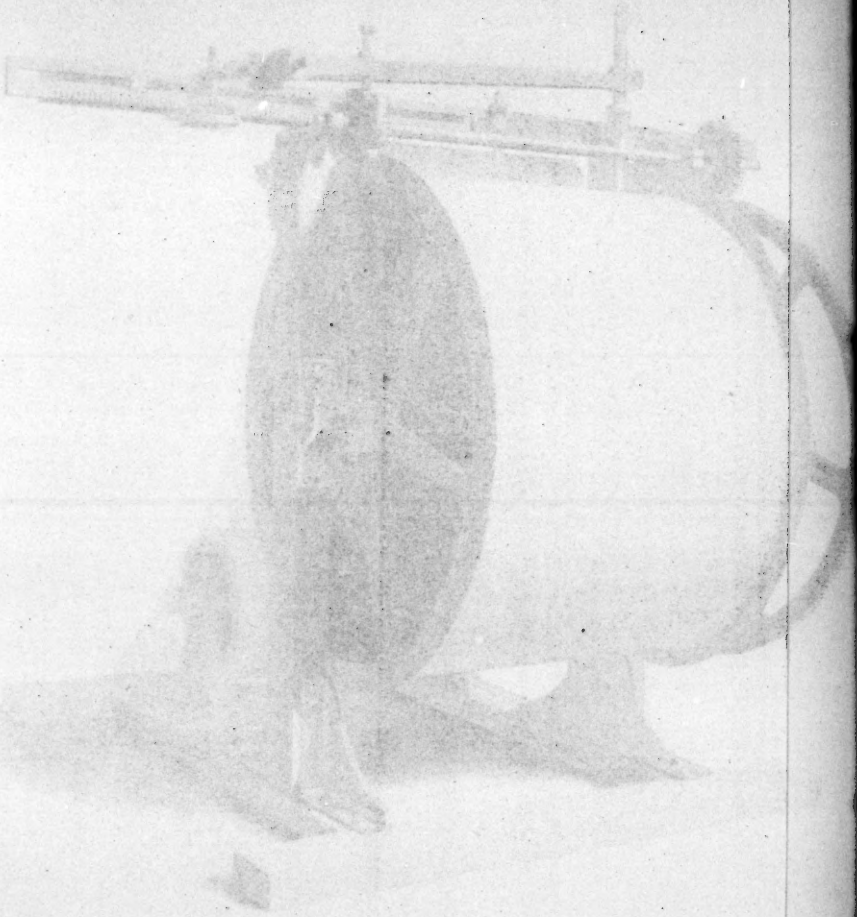
N°1.

Phil. Trans. 1831. Page 209.



Perspective View of a Graphic Registrer of the Tides and Winds.

Engd. by T. Bradley.



N°11.

A *Roller on which the Paper is wound*

B *The Drawing Cylinder*

C *Roller for keeping the Paper in contact with the Drawing Cylinder*

D *Toothed wheel connected with the Clock*

E *Camb Wheel for raising the Hammer*

F *The Hammer*

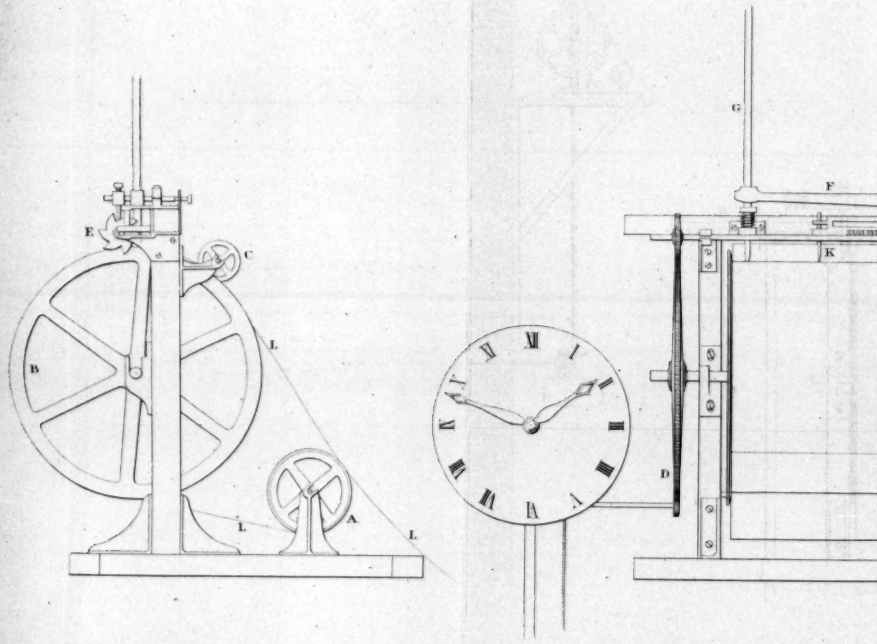
G *The Spindle with the Arrow upon it which is connected*

H *The Rack*

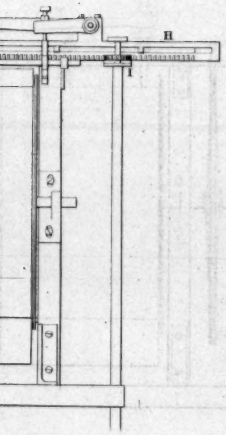
I *The Pinion connected with the Float*

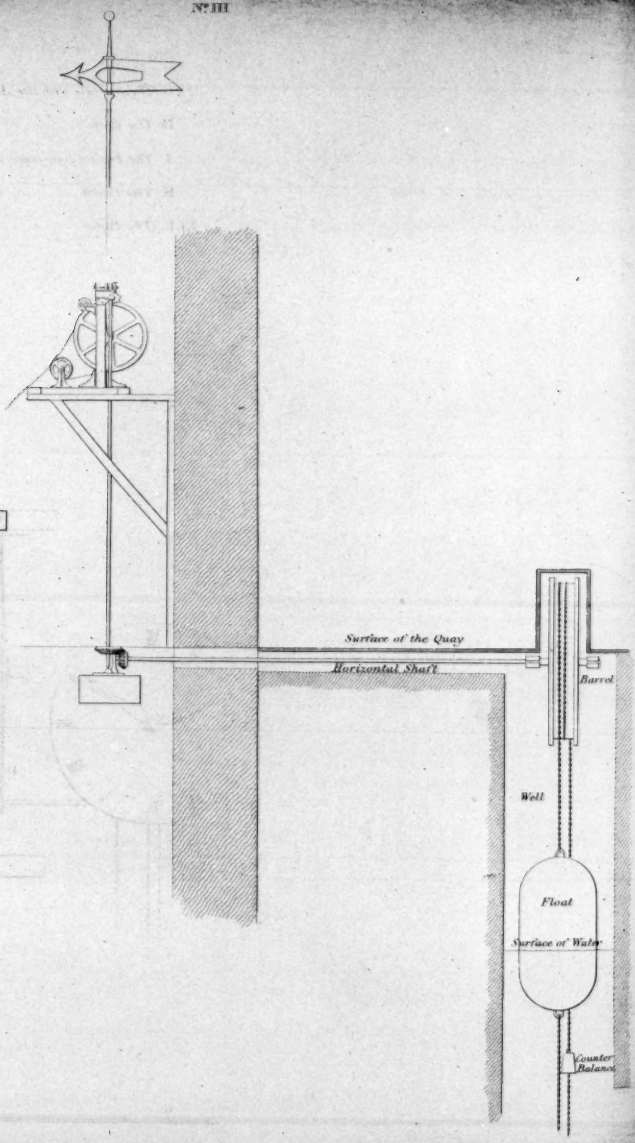
K *The Pencil*

LLL *The Paper*

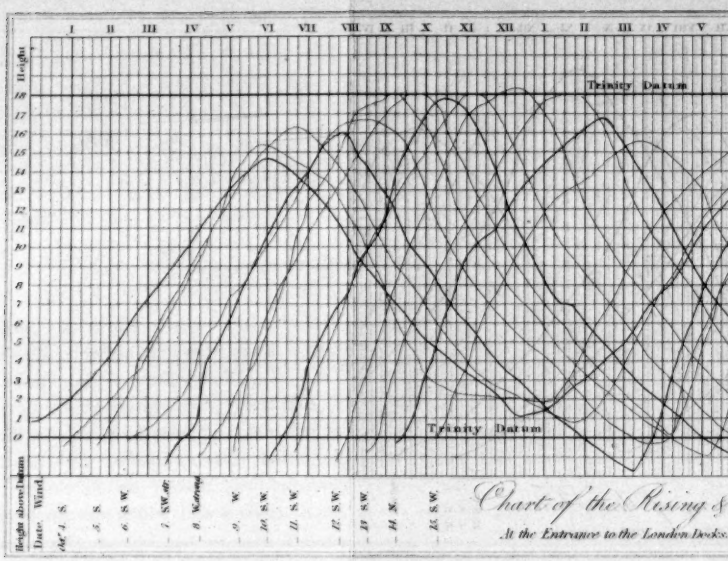


cted with the Weathercock

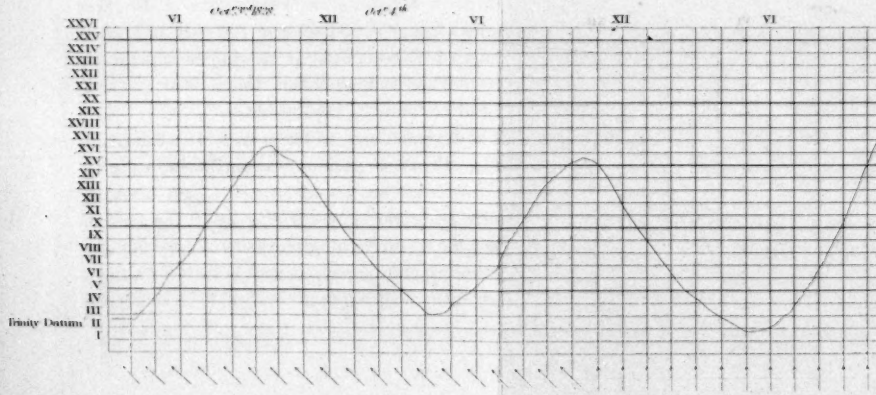


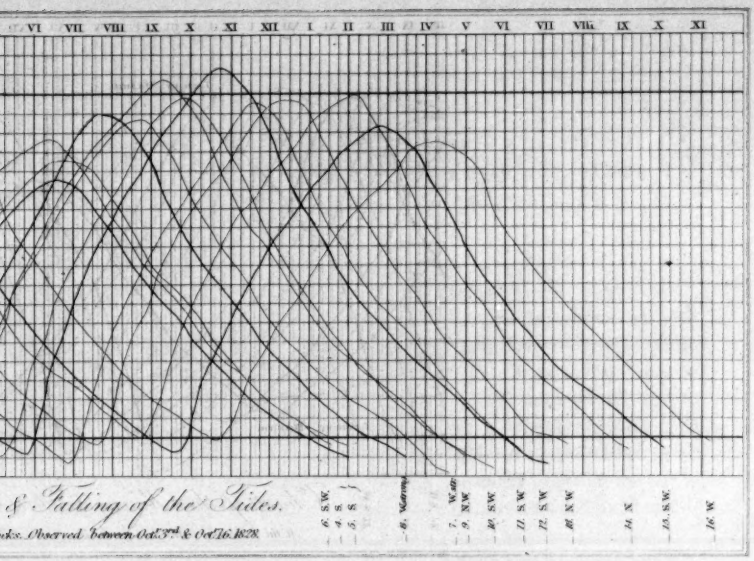


Nº IV


I II III IV V VI VII VIII IX X XI XII I II III IV V
Height
18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
Trinity Datum
Trinity Datum
Height above Datum
Date. Wind.
Oct. 4. S.
5. S.
6. S.W.
7. S.W. & W.
8. W. & S.
9. W.
10. S.W.
11. S.W.
12. S.W.
13. S.W.
14. S.W.
15. S.
16. S.W.
Chart of the Rising &
At the Entrance to the London Docks.

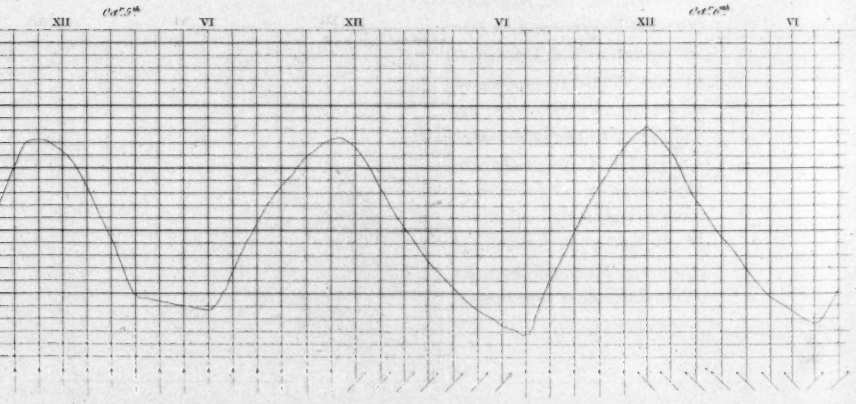
Nº V


VI
XII
VI
XII
VI
XXVI XXV XXIV XXIII XXII XXI XX XIX XVIII XVII XVI XV XIV XIII XII XI X IX VIII VII VI V IV III
Trinity Datum II
I



& Falling of the Tides.

ks. Observed between Oct.3rd & Oct.16. 1829.



Engd by T. Bradley

XII. *Description of a Graphical Registrer of Tides and Winds.* By Henry R. Palmer, *Civil Engineer.* *Communicated by* John William Lubbock, *Esq. V. P. and Treas. R.S.*

Read March 10, 1831.

In the exercise of my profession, particularly in reference to questions relating to harbours and tidal rivers, I have frequently had occasion for a train of observations on the rate of the rising and falling of the tides, continued through at least one whole series of them. Such observations have usually been made at intervals of fifteen minutes, and in order to exhibit the rates of rising and falling in the different periods of the same tide, and the differences of the several tides, I have represented them in the form of a chart.

The Plate No. IV. exhibits one of those charts which was made previously to the commencement of a new entrance to the London Docks. The use of it was to ascertain for what length of time there would be a given depth of water about the said entrance, and in the channel of the river near to it, during the different tides.

Having for some time past directed my attention to the nature of the effect that will be produced on the tides of the river Thames in the port of London by the removal of London Bridge, I have thought it very important that the changes should be progressively marked as they occur, that the ultimate consequences may with more ease and certainty be demonstrated.

In order to continue a series of constant, and, as far as possible, of unerring observations through so great a length of time as must necessarily elapse during the progress of these changes, I have realised an idea which I have for many years entertained, and for which the works now carrying on at the London Docks, under my direction, afford me an excellent opportunity of carrying into effect, viz. To construct a machine which being acted upon jointly by a time-piece, and a float resting on the water, shall represent every tide in succession

by such lines as those in the chart before referred to; while by connecting a type in the machine, with a weather-cock, above the house in which it is placed, an hourly register of the winds would also be obtained.

The performance of such a machine must if well arranged be evidently free from those inaccuracies and doubts which the frequent and long-continued observations of individuals, through nights as well as days, must be liable to. It will require only the occasional attention of a superintendant to correct the time, and supply it with paper.

The following is a description of the first that has been made for me, and which will shortly be placed at the mouth of the new entrance to the London Docks.

Reference to the Plates.

No. 1. A perspective view of the machine.

No. 2. An elevation and side view.

No. 3. The well and float, with the relative situation of the machine.

No. 4. A chart of tides in the river Thames observed in 1828.

No. 5. The same tides represented according to the form to be effected by the machine.

A number of parallel and equidistant lines, representing feet in height, are engraved, and printed on dry paper, whose sides are carefully cut parallel, and the ends joined until such a length is formed as to serve for three or four weeks consumption.

The long sheet thus produced is wound upon a brass roller, which is placed near the lower part of a cylinder one foot in diameter, so that the paper may pass round that cylinder, and be in contact with it through about three fourths of its circumference.

The contact of the paper is preserved by a roller pressing upon it by its own weight near the upper part of the cylinder.

On the axis of the cylinder is a toothed wheel, which is to be acted upon by a clock, and hence follows the motion of the paper in the direction of its length, moving equal distances in equal times. By means of the same toothed wheel, motion is given to a Cam Wheel having six teeth, and the velocity so regulated that it makes one revolution in six hours. Each tooth in the Cam Wheel

raises, in its turn, a hammer, whose fall strikes an impression on the paper once in every hour, so that the spaces passed through are measured as they occur, and leave no error that might otherwise arise from the expansion or contraction of the materials. It is obvious that when the paper has been taken off the machine, vertical lines must be ruled across the paper through the hourly points made by the hammer, and any of them may be subdivided into less measures of time if required. Immediately over the axis of the cylinder, and parallel with it, is a rack, which carries a pencil, and is acted upon by a pinion, which receives its motion from a float resting upon the water; so that as the tide rises and falls, the pencil moves backwards and forwards at a proportional rate, although through diminished space, and thus by the combined motions of the clock and the tide, a line is produced on the paper which represents both.

The impression hourly struck by the hammer is the figure of an arrow with a cross in the centre, the point of intersection being that through which the hour lines are drawn. The figure is cut on the end of an upright punch, which is connected with a weather-cock on the top of the house in which the machine is placed, and hence the direction of the arrow will always correspond with that of the wind, which becomes hourly registered.

The marking point for the tidal line is of steel, which makes such an impression on the paper as will be easily traced with a camel's hair pencil when the paper is removed, as will also the impression made by the wind arrow. A pencil, usually called the metallic pencil, may be employed for the original line if preferred.

The float which rests on the water is a hollow plate-iron vessel, suspended by a chain which passes twice round a light cast-iron barrel, and then descends, having a counter weight attached to it. The chain is of such a length, that both ends of it are always resting on the ground, so that the weight of chain on each side of the barrel is always equal.

The float is placed in a well communicating with the river, and to prevent that undulating motion which would be produced by the motion of the surface of the river during high winds, the water is received into the well through a fine wire gauze.

The motion is communicated from the float in the well to the machine in

the house by a light horizontal shaft revolving under the surface of the ground, and communicating by bevil wheels with an upright shaft which acts immediately upon the pencil rack *.

The references to the details are given on the several Plates.

Considering how many interests may be affected by the expected change in the circumstances of the river Thames, by the removal of London Bridge, it appears important that a similar gauge should be established at some point *above* the bridge, so that by the corresponding charts of the two machines, the facts will be so clearly exhibited as to defy that contradictory evidence which interested persons might hereafter produce, as well as furnish valuable data upon which remedial expedients might be founded.

Since the completion of the machine above described, my attention has been directed by Mr. Lubbock to the application of similar means for observing accurately the times of high water, not only as a scientific inquiry upon the tides generally, but with a view to the construction of accurate tide tables. Perceiving the value of that gentleman's suggestion, I devoted immediate attention to the subject, and trust, from the progress I have made, that the object will be accomplished in a satisfactory manner.

To indicate sensibly the time of high water to within any small portion of time (such as a minute), required a representation on a much larger scale than that above described, and therefore a distinct cylinder became necessary. It not being required to register the absolute heights by this machine, I have determined on registering only the upper part of the tide, that is, that portion which is contained in thirty minutes, of which the highest point will be one. Instead of representing the upper part of the tide by a continuous line, it will indicate the relative place and the time of it by punctures, one of which will be struck in each minute; so that the real time of any one puncture being known, the observer has only to count the number of punctures from that whose time is known, to the highest on the scale, and that will denote the time sought.

To accomplish this, the instrument by which the marks are made on the paper will not move as the pencil does in the first machine, but, keeping their

* To avoid derangement of the rate of the clock's motion, a weight is to be attached to the registrer, equal in force to the resistance of the friction of those parts the clock has to move.

places, the cylinder with the paper will move, and thus by a very simple motion from the clock, the real time will be expressed once in the thirty minutes around one of the minute points.

To avoid the risk of inattention on the part of the individual to whom the operation of the machine may be intrusted, the machinery connected with the float will put the registering apparatus into gear a short time before the time of high water, and the clock will put it out of gear when the operation has been continued for thirty minutes.

To obtain the exact time of high water, or even to ascertain it with moderate precision, does not appear practicable but by means similar to those described. The motion of the surface of the water, occasioned by the wind, and the vibrating motion of the general mass upon the turn of the tide, are obvious reasons: but by having the float placed in a protected situation, and the varying height *marked in fact by the time*, we may expect to arrive at a sufficient accuracy for the purposes required.

When this machine has been completed, a full description of it shall be forwarded to the Royal Society, and I hope it will hereafter do me the honour to accept either the original or copies of the series of observations upon the tides of the Thames, which both machines may exhibit.

XIII. *On the Errors in the Course of Vessels, occasioned by Local Attraction; with some Remarks on the recent Loss of His Majesty's ship Thetis.* By Peter Barlow, *Esq. F.R.S. Cor. Mem. of the Inst. France, Imp. Acad. Sciences St. Petersburgh, Acad. Sciences Brussels, &c. &c.*

Read April 21, 1831.

On presenting the following remarks to the attention of the Royal Society, I feel it necessary, first, to apologize for their not possessing that degree of scientific novelty which is generally expected in such communications,—and secondly, that in enforcing my argument, I may perhaps seem to give more value to my own investigations than is consistent with good taste. I must, however, either in some measure do this, or leave the evil untouched, which I feel it my duty to endeavour to remove; and therefore, trusting to a liberal interpretation of my motives, I shall state without reserve such facts as appear necessary to establish the object I have in view, and it is hoped the want of novelty will be compensated by the importance of the subject under consideration.

That a ship's compass is subject to a deviation from its true direction in consequence of the attraction of the iron used in the construction and appointments of the vessel, is now too generally admitted to require any argument, although I believe there may be still some few officers who are sceptical on this point; I have at least been seriously assured by one of rank and long standing, "that there certainly was no local attraction when he was at sea." Now there is really more in this observation than one would imagine, for there can be no question that forty years back the error arising from this disturbing force was very inconsiderable to what it is at present; every year in fact increasing the amount, and rendering a correction of the error more and more necessary. This increase is occasioned by the immense quantity of iron now employed in the construction of a ship of war and its appointments. At the period above

alluded to, iron ballast and iron tanks were perhaps scarcely known; now we have, besides these, iron knees, iron cables, and above all iron capstans, besides various other articles of the same material, which together form such an attracting mass, that if we cannot allow that there was no local attraction forty years back, we must at least admit that it was certainly then very inconsiderable to what it is at present, and that navigation by compass was at that time comparatively on equal terms with nautical astronomy; but since that period, the errors of the one have been gradually removed by improvements in instruments, the introduction of chronometers, and the correction of astronomical tables and data, whereas the compass still remains the same uncouth machine, and the disturbing forces to which it is exposed have been increased in, perhaps, a fourfold proportion. At all events the disturbing force is now considerable, and the deflection it causes in the needle, under some circumstances, very great; and as this effect is perpetually varying as the course of the vessel is changed, as it is also as she changes her latitude, though the course should remain the same, a constant attention to the amount of this error seems to be indispensable, at least in those circumstances where the whole safety of the vessel is dependent on the certainty of the courses steered, which is in fact always the case in a dark night, and when land is near.

It is almost impossible to give a very popular idea of the direction and amount of this deflecting force. It may however be stated, that in this latitude, and in all those northern latitudes where the dip is considerable, the greatest deflections take place, on an east or on a west course, diminishing both ways to the north and south, where it vanishes; and in all these cases the direction of the deflection is always to the right or left of a person, looking forward in the vessel, accordingly as the course of the vessel is to the right or left, that is to the east or west of the meridian, and it is exactly the reverse in a high southern latitude or with a considerable southern dip. But as we approach the equator, where the dip is small, the deflections at the east and west points vanish, and we have then four points of greatest attraction, viz. the N.E., N.W., S.E., and S.W., the direction of the deflection changing as we pass through the N., S., E., and West points; but the deflection at its maximum is much less in these latitudes than in those where the dip is more considerable, all other things being the same.

These rules, which are given upon the supposition that the compass is situated in its usual place, aft in the vessel, will furnish a general idea of the direction of the deflection. With regard to the actual amount, it is of course different in different vessels, varying in these latitudes from 5° to 12° or 14° with an easterly or westerly course; which become greater as we increase our latitude, but diminish (without however vanishing) at the equator, whence it again increases as we approach the southern pole.

The following are some results, several of which I have assisted in taking; they rest, as will be seen, on the best authorities, and will give a good general idea of the maximum amount.

Ship.	Commander.	Place.	Local Attraction.
			° ′
Conway . .	Captain Basil Hall . .	Portsmouth . .	4 32
Leven . .	Captain Owen	Northfleet . .	6 7
Baracouter .	Captain Cutfield . . .	Northfleet . .	14 30
Hecla . .	Captain Sir E. Parry .	Northfleet . .	7 27
Fury . . .	Captain Hopner . . .	Northfleet . .	6 22
Griper . .	Captain Clavering . .	Nore	13 36
Adventure .	Captain King	Plymouth . .	7 48
Gloucester .	Captain Stuart . . .	Channel . . .	9 30

Giving a mean of 8° 44′ at the east and west points in these latitudes.

The latter observation by Captain Stuart is from the remark-book of the Gloucester, and from which I beg to give the following extract:

"1830. 30th August. From not having had a favourable opportunity of ascertaining the ship's magnetism or local attraction in steering down the British Channel, I only allowed the true variation as found, but observed the ship was invariably drawn to the southward of her intended place, notwithstanding the greatest care being taken in steering her. But on taking an amplitude of the sun at setting on the 1st September, I found the variation to be 34° W. (when the ship's head was west), which difference from the true variation in the Channel 24° 30′ W. will account for the ship being so drawn to the southward of her intended track."

It would be quite superfluous to give further evidence of the existence of

this disturbing power; and little, I conceive, need be said to show how much such deviations from the estimated course of a vessel, in channels and narrow seas, are calculated to lead to the most disastrous events. To take the last case, for example, where the deviation is 9° 30′, and for the deviation in miles the general expression (dist. × 2 sin ½ deviation), we shall find that after running ten miles, the vessel would be more than a mile and a half to the southward of her reckoning; in a distance of twenty miles, three miles and a quarter to the southward; in thirty miles, five miles to the southward, and so on as the distance increases.

Now it requires no knowledge of navigation to estimate the fatal consequences that might attend such an error in a narrow channel and in a dark night, if it were wholly unknown or disregarded. We see also how very easy it is, after an accident has occurred, to imagine a current (unknown of course to exist before), to account for the disaster. The Gloucester, for example, in the above instance was constantly "drawn to the southward;" and this might have been set down to the effect of a current, had it not been proved to be local attraction.

That a ship is sometimes involved in an unknown or unusual current, which may lead her into an error of reckoning, no one can for a moment deny; but I do at the same time maintain, that unless a proper attention be paid to the local attraction, a vessel is as it were in a perpetual current, setting sometimes in one direction, and sometimes in another, sufficient to baffle the most experienced pilot; and I further maintain, that science and humanity both require, before we admit the plea of unknown currents to explain the cause of every disaster, that it be sufficiently ascertained how far allowance has been made, or a correction obtained for that current, which it is now well known a vessel carries with her through every league of her voyage.

Let us now turn to the late melancholy wreck of His Majesty's ship Thetis. It appears from the account given of this disaster in the United Service Journal, that "the Thetis sailed from Rio Janeiro on the 4th of December, with a million of dollars on board, besides other treasure, and every prospect of a fine passage, stretching away to the S.E. The next day, the wind coming rather favourable, they tacked, thinking themselves clear of land; and so confident were they, that the top-mast studding sails were ordered to be set, the ship run-

ning at the rate of nine knots; and the first intimation they had of being near land, was the jib-boom striking against a high perpendicular cliff, when the bowsprit broke short off, the shock sending all three masts over the side;" and thus in a moment perished twenty-five valuable lives, and a fine vessel, with her cargo, worth nearly a quarter of a million sterling.

Here then we have a case of a ship leaving port one day, with every prospect of a fine passage, which had so far lost her reckoning on the evening of the next day, as to be wrecked on a rock not more than seventy miles from her point of departure, which was supposed to be some miles to her west.

I have no desire to prejudge the cause of this unfortunate misreckoning; I wish only the true cause should be ascertained. In the letter of the commander of the Thetis to the admiral on the station, he says, that "from all the precautionary measures taken, nothing but the strongest currents, and the thick hazy weather, and hard rain, can be pleaded in extenuation."

I most sincerely hope that amongst "the precautionary measures taken," that of correcting or making proper allowance for the local attraction was included; for I have no hesitation in asserting, if such precaution was not taken, that this omission would be quite sufficient to account for the accident. It is obvious from the general principles which I have stated in the preceding part of this paper, that at Rio, where the dip of the needle is about 22° south, (the course steered on the 4th of December having been S.E., and on the 5th of December necessarily somewhere between the east and north,) the local attraction would be constantly drawing the vessel over to the westward, and there can be no question that her being more to the westward than her reckoning, was, from whatever source it might have proceeded, the cause of the disastrous event.

Without therefore in any way prejudging the case, I have only to express a hope that some inquiry may be made, to ascertain whether any and what allowance or correction was made for the local attraction of the vessel.

It is impossible now, if the local attraction of the Thetis has not been before taken, to know its amount; but we can ascertain, at least approximatively, what would have been the deflection of the Gloucester under similar circumstances; and as the amount of attraction in this vessel is nearly the mean of all those I have given in a former page, it may be interesting to see the result.

It has been stated that the maximum attraction or deflection is much less (all other things being the same) with a small dip than with a large one; the proportion being, "that the tangents of the angles of deflections are inversely as the cosines of the dip:" now, the dip at Rio Janeiro being about 22°, and in London 69½°, we have

$$\cos 22^\circ : \cos 69\tfrac{1}{2}^\circ :: \tan 9^\circ\ 30' : \tan 3^\circ\ 33'.$$

That is, the Gloucester leaving Rio under similar circumstances to the Thetis, would on any course about the S.E. or N.E. (the former being stated as the course of the latter vessel on the 4th, and the other her probable course after tacking on the 5th of December), be constantly deflected about 3½°* out of her supposed course, and as the sine of 3° 33′ is about $\frac{1}{16}$th part of the radius, it is obvious, taking only the error due to the 5th of December, and reckoning the distance run at eighty miles, that the ship would pass five miles nearer to Cape Frio than her reckoning, an error quite sufficient to account for the fatal catastrophe which has occurred to the Thetis; for it appears that a distance of only so many fathoms would have nearly carried her clear of the land. I do not include the error due to the first day, because its tendency would only be to carry the vessel to the southward about the same quantity, which of itself could have produced no evil.

After all, let it be remembered that this is a supposititious case, and that my object in stating it is merely to show what might happen if the deflection from local attraction were disregarded, and thereby to prove the propriety and necessity of ascertaining whether in the case of the Thetis, and in all similar cases, the proper correction was made, before the apology of currents can be admitted.

I urge this the more particularly, because I fear this source of error is too much disregarded, and as I think it probable that in the several investigations which have been held to inquire into cases of vessels lost in a similar way for the last ten years, since I have been interested on this subject, no question has been asked whether or not the error of the compass had been corrected; and thus vessel after vessel is lost, and current after current is imagined to ac-

* I say about 3½°, because much depends upon the direction of the centre of attraction in the vessel.

count for the loss, while an actually existing and known cause is allowed to remain uncorrected and disregarded.

That the remedy for this evil, which I have been so fortunate as to discover, is simple and universal, is, I believe, generally admitted; indeed, after being submitted to trial by two of our most scientific officers from 57° south latitude to 80° north latitude, and having been found to be effective to the most extreme point, it is impossible that any doubt should remain on that head.

It must be, therefore, that the error itself is disregarded, and it would consequently be rendering an essential service to the navy, when any loss is sustained from neglecting this necessary precaution, that it should be traced to its proper source; and it is with this view that I have drawn together these few remarks. If I overrate the importance of the error or the value of the remedy, my apology must be the opinions which have been given on the subject by many distinguished naval commanders, both English and foreign, and the high marks of approbation with which my investigations have been acknowledged by the Royal Society, and other learned societies of Europe. To which I may also add my anxiety, that where science can be brought to facilitate the progress of navigation, and to contribute to its security, it may not be allowed to be neglected in the British navy.

XIV. *On the Meteorological Observations made at the Apartments of the Royal Society during the Years* 1827, 1828, *and* 1829. *By* J. W. Lubbock, *Esq. V.P. & Treas. R.S.*

Read April 14, 1831.

THE phenomena which principally deserve attention connected with the science of meteorology, are:

1. The annual and diurnal variations of the barometer and thermometer, due to the action of the sun.

2. The variations of the barometer due to the moon, and dependent on her age.

3. The comparative temperature and barometrical pressure at different points of the earth's surface, the isothermal lines, and lines of equal barometrical pressure.

4. The influence of the direction of the wind on the temperature and barometrical pressure.

5. Phenomena connected with the electrical state of the air, the aurora borealis, &c.

In order to determine the annual variations of the barometer, I have taken the mean of the observations in each month, made at the apartments of the Royal Society, during the years 1827, 1828, 1829, and 1830. The results are given in the following Table, which shows the differences from the mean*.

The two first columns result from these observations, reduced to 32° Fahr., and corrected for capillarity.

The four other columns are deduced from Table 3 in the valuable work of M. Bouvard "Sur les Observations Météorologiques." (Mémoires de l'Academie des Sciences. Vol. vii. p. 312.)

* The mean result being given for each year separately in the Philosophical Transactions, of course it was only necessary for me to add these together, and take the fourth. Since the reading of the paper, the observations of 1830 have been added and taken into account.

	Obsns at Somerset House.		Obsns at the Paris Observatory.			
	9 A.M.	3 P.M.	9 A.M.	12.	3 P.M.	9 P.M.
January	+.006	+.005	+.085	+.067	+.072	+.069
February ..	+.064	+.070	+.071	+.070	+.065	+.063
March	−.004	−.009	−.006	−.004	−.007	−.005
April	−.044	−.143	−.044	−.047	−.053	−.042
May	+.002	−.025	−.044	−.043	−.047	−.046
June	+.006	−.031	+.037	+.040	+.040	+.064
July	−.017	−.022	+.008	+.004	+.009	+.008
August	−.005	−.001	+.018	+.016	+.014	+.013
September ..	−.039	−.048	+.016	+.014	+.015	+.019
October	+.117	+.116	−.063	−.060	−.062	−.056
November ..	+.036	+.025	−.021	−.015	−.012	−.011
December ..	−.006	+.005	−.047	−.042	−.035	−.039
Mean	29.861	29.840	29.778	29.767	29.748	29.762

Thus the mean height of the barometer at 9 A.M. for January is 29.861 + .006 = 29.867.

It may be remarked, that according to this Table, the annual variations appear to be independent of the diurnal variations. The Paris observations present much greater regularity than those made here, which results perhaps from their greater number. In order to determine the diurnal variations of the barometer, it is necessary that the observations should be repeated much more frequently in the course of the day than is done here at present. The mean height of the barometer here at 9 in the morning is greater by .021 inch (or about $\frac{1}{50}$th of an inch), than at 3 in the afternoon; and so regular is this diurnal variation, that considering the mean of each month separately for the years 1827, 1828, 1829, and 1830, there are only two cases in which the mean height is greater at 3 in the afternoon than at 9 in the morning. The corresponding difference at Paris is .030 inch *.

In order to determine the fluctuations of the barometer due to the moon, it would have been desirable to possess many more observations; but, unfortunately, previous to 1827, the observations of the barometer at Somerset House seem not to have been made at stated times of the day, a condition which appears to me absolutely necessary, in order that meteorological observations

* There is a very interesting paper on the annual and diurnal variations of the barometer, by M. Carlini, in the 20th volume of the Memorie della Società Italiana. Fasc. 1mo. (Memorie di Matematica.)

may be applied to this or any other useful purpose, except that of serving at the time to prognosticate the weather, or but imperfectly to determine the correction due to the direction of the wind.

I was therefore obliged to confine myself to the years 1827, 1828, and 1829. The method which I have adopted with respect to these aerial tides is similar to that which I have used in order to determine the phenomena of the tides in the river Thames, and consists in classifying all the heights of the barometer, and taking their mean, which correspond to a particular age of the moon, defined by the circumstance of her transit taking place in a given half-hour of the day. Thus all the days in the years 1827, 1828, and 1829, were found when the moon passed the meridian between 12 and half past 12, and the mean of the transits taken, which of course is nearly a quarter past 12; the heights of the barometer were then taken on the same days, and the mean taken; and thus all the transits of the moon which occurred during the years 1827, 1828, and 1829, were taken, and the corresponding observed heights of the barometer selected and compared with them. The height of the attached thermometer was also taken, and the mean height of the barometer corrected afterwards by the mean height of the attached thermometer, so as to reduce it to 32° Fahr.

Although the transits of the moon were at first classed for every half-hour, I afterwards combined them for every hour, in order to make use of a greater number of observations in obtaining results. The mean transit thus found, scarcely differed from the half-hour, which is therefore taken as the time of the moon's transit in the following Table, in which the results are exhibited.

TABLE.

No. of Obs.	Time of Moon's transit.	Moon's age.	9 o'clock A.M. Barom.	Attach. Therm.	Barom. red. to 32° Fahr.	Diff. of Barom. from mean.	3 o'clock P.M. Barom.	Attach. Therm.	Barom. red. to 32° Fahr.	Diff. of Barom. from mean.
	h. m.	days.	inches.		inches.	inch.	inches.		inches.	inch.
85	0 30 or 12 30	.6 or 13.3	29.983	54.3	29.926	+.063	29.956	55.9	29.894	+.055
87	1 30 or 13 30	1.8 or 16.6	29.951	55.5	29.891	+.028	29.913	57.4	29.859	+.010
90	2 30 or 14 30	3.0 or 17.8	29.916	54.0	29.860	−.003	29.898	56.0	29.836	−.003
89	3 30 or 15 30	4.3 or 19.1	29.892	53.8	29.836	−.027	29.883	55.5	29.823	−.016
88	4 30 or 16 30	5.5 or 20.3	29.904	53.3	29.849	−.014	29.891	54.8	29.832	−.007
92	5 30 or 17 30	6.8 or 21.5	9.896	53.8	29.840	−.023	29.892	56.1	29.830	−.009
86	6 30 or 18 30	7.9 or 22.7	29.924	54.0	29.868	+.005	29.897	56.1	29.835	−.004
93	7 30 or 19 30	10.4 or 25.2	29.899	53.9	29.843	−.020	29.884	55.7	29.823	−.016
87	8 30 or 20 30	10.4 or 25.2	29.890	53.9	29.834	−.029	29.870	55.6	29.809	−.030
86	9 30 or 21 30	11.7 or 26.4	29.910	53.6	29.855	−.008	29.901	55.8	29.839	.000
88	10 30 or 22 30	12.9 or 27.7	29.911	52.9	29.857	−.006	29.913	55.2	29.854	+.015
88	11 30 or 23 30	14.2 or 28.9	29.938	53.1	29.874	+.011	29.919	55.2	29.860	+.021
	Mean		29.919	53.9	29.863		29.901	55.8	29.839	

The following Table results from Table VI. of M. Bouvard, (p. 316,) reduced to English feet.

	Variation of Barometer from mean.	
	9 A.M. inch.	3 P.M. inch.
Day of the Syzygy	−.004	−.008
First day after the Syzygy	−.010	−.006
Second day after the Syzygy	−.013	−.009
Second day before Quadrature	+.008	+.005
First day before Quadrature	+.024	+.032
Day of the Quadrature	+.025	+.017
First day after the Quadrature	−.002	−.001
Second day after the Quadrature	−.000	+.012
Second day before the Syzygy	−.019	−.018
First day before the Syzygy	−.009	−.019
Mean height	29.781	29.748

The results afforded by the observations at Somerset House differ widely from those above obtained by M. Bouvard from the observations at the Paris Observatory; according to the former, the barometer is highest at new and full moon and lowest at quadrature; according to the latter, the contrary is the case.

The extent of the fluctuations of the barometer due to the moon according to the former is about .08 or nearly $\frac{1}{10}$th of an inch, according to the latter only ·05 or $\frac{1}{20}$th of an inch.

They agree in this, that the fluctuations take place nearly in the same manner in the morning and in the afternoon; whence it follows, that the period of the principal inequality of the height of the barometer due to the action of the moon is not the same as that of the ocean; for if it were so, as the observations are made at a distance of six hours, the maximum in the morning would correspond to the minimum in the afternoon.

Laplace enumerates among the most important causes of the fluctuation of the pressure of the atmosphere, the rising and falling of the ocean due to the action of the sun and moon, the ocean serving as the basis or support of the atmosphere. But with that deference which is due to the authority of so great

a mathematician, I must confess that this cause does not appear to me adequate to produce any sensible effect; for in the open sea the variations of the height of the water due to the tides, where this cause would be most felt, do not exceed three or four feet, and any considerable rise of the tide is in general confined within very narrow limits, as in channels and between the banks of rivers. Lastly, I have endeavoured to ascertain how far the barometer is affected by the direction of the wind; and the following Table gives the results which I have obtained with this view. The fluctuation due to this is much greater than that due to any other cause; and it is therefore very important that this correction should be carefully ascertained, in order that it may be applied when observations of the barometer are classed, in order to determine any other inequality. The barometer is lowest, as might be expected, in the rainy quarters, as S.W. and W.S.W.

Table showing results deduced from the Meteorological Observations made at Somerset House during the years 1827, 1828, and 1829, classed according to the direction of the Wind.

Direction of the Wind at 9 A.M.	No. of Observations.	9 o'clock A.M.		3 o'clock P.M.		Dew point at 9 A.M. in degrees of Fahr.	Rain in inches, read off at 9 A.M.
		Barometer.	Attach. Therm.	Barometer.	Attach. Therm.		
N	117	30.009	47.4	30.014	49.3	41	.019
NNE	88	30.068	52.2	30.043	54.0	45	.032
NE	35	29.923	49.3	29.912	51.1	42	
ENE	16	30.005	49.4	29.941	52.8	45	.100
E	74	29.915	53.2	29.889	55.8	47	.021
ESE	68	29.951	52.5	29.916	54.2	45	.022
SE	30	29.793	57.8	29.772	59.2	52	.053
SSE	39	29.742	55.8	29.680	57.9	51	.052
S	73	29.808	54.4	29.795	56.4	49	.033
SSW	88	29.815	56.7	29.782	58.5	51	.038
SW	103	29.884	54.8	29.835	56.6	48	.069
WSW	143	29.900	54.5	29.898	56.7	48	.059
W	83	29.944	57.0	29.936	59.1	49	.059
WNW	18	29.889	57.2	29.887	60.0	50	.053
NW	65	29.936	55.6	29.802	57.4	46	.054
NNW	56	29.978	54.8	30.014	56.8	46	.080
Mean of Total		29.918	54.7	29.893	55.7	47	.044

TABLE deduced from the preceding, showing the variations of the Barometer reduced to 32° FAHR.

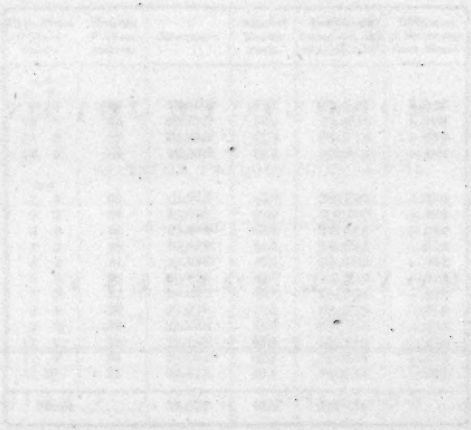
Direction of the Wind.	No. of Observations.	9 A.M.	3 P.M.
N	117	+.114	+.140
NNE	88	+.158	+.155
NE	35	+.021	+.032
ENE	16	+.100	+.056
E	74	−.002	−.005
ESE	68	+.038	+.029
SE	30	−.128	−.128
SSE	39	−.178	−.218
S	73	−.107	−.098
SSW	88	−.106	−.113
SW	103	−.034	−.056
WSW	143	−.015	+.003
W	83	+.023	+.036
WNW	18	−.032	+.013
NW	65	+.013	−.093
NNW	56	+.018	+.118
Mean		29.863	29.835

I shall not attempt to enter into any discussion of the influence of electrical phenomena upon the weather; no observations with reference to this part of the subject have yet been made here.

I have to acknowledge the very kind assistance of Mr. DEACON, (to whom I have been indebted before,) in forming the Tables which accompany this paper. I have not discussed the circumstances under which the observations have been made which serve for the foundation of the results which are here presented, although I fear that the instruments employed are unworthy of the Society and of the care bestowed upon the observations by Mr. HUDSON. This discussion would have been necessary if my object had been to determine the mean temperature or the mean barometrical pressure at London; but as I have only endeavoured to ascertain the fluctuations of the barometer due to certain causes, whose periods are independent of any errors that may arise from the construction and condition of the instruments, those errors are of little importance in the preceding investigation.

Since this paper was read, Mr. Hudson has made some observations with a view to determine the diurnal variation of the barometer; they were begun on the 26th of April, and have been continued to the present time, June 13th. The results are exhibited in the following Table, which seem to indicate a minimum about 6 o'clock P.M.

Mean Times of Observation.	Number of Observations.	Barometer.	Attached Thermometer.	Barometer corrected, and reduced to 32°.	Difference of Barometer from Mean.
A.M. h. m.					
9 0	49	29.900	61.4	29.821	+.025
10 4	43	29.885	62.8	29.803	+.008
11 2	47	29.895	62.4	29.814	+.019
12 3	38	29.902	63.3	29.819	+.024
P.M.					
1 4	43	29.866	63.5	29.783	−.012
2 3	44	29.887	63.8	29.803	+.008
3 1	49	29.880	63.4	29.797	+.002
4 3	40	29.868	64.1	29.783	−.012
5 2	44	29.856	63.5	29.773	−.022
6 4	39	29.837	63.6	29.753	−.042
7 3	37	29.877	63.2	29.794	−.001
8 4	36	29.858	61.5	29.779	−.016
9 3	38	29.863	61.0	29.785	−.010
10 2	33	29.896	60.8	29.819	+.024
11 2	34	29.884	60.9	29.806	+.010
11 57	34	29.874	60.4	29.790	−.005
Mean		29.877	62.6	29.795	



METEOROLOGICAL JOURNAL,

KEPT BY THE ASSISTANT SECRETARY

AT THE APARTMENTS OF THE

ROYAL SOCIETY,

BY ORDER OF

THE PRESIDENT AND COUNCIL.

MDCCCXXXI.

METEOROLOGICAL JOURNAL FOR JULY, 1830.

1830. July.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	Fahrenheit. 3 P.M.	Self-registering. Lowest.	Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
♃ 1	29.943	71.8	29.809	71.4	60	68.3	66.6	53.7	69.7		ESE	Fine—cloudy—light brisk wind.
♀ 2	29.728	69.0	29.656	69.2	61	65.4	67.4	53.8	67.6	0.333	SSE	Fine—light clouds—showery—light brisk wind.
♄ 3	29.484	70.4	29.580	68.3	54	62.5	66.0	56.3	68.7	0.156	NNE	Fair—cloudy—light wind. Thunder and lightning with rain at 1 P. M.
☉ 4	29.788	66.7	29.890	69.7	58	63.3	71.8	55.7	72.7	0.181	W	Fine and clear—cloudy.
☽ 5	30.003	64.3	30.045	68.0	56	59.6	66.2	55.3	67.7		NNE	Overcast—light wind.
○ ♂ 6	30.029	70.7	29.930	70.0	57	66.8	66.8	57.3	69.3		SSW	Cloudy—brisk wind.
☿ 7	29.654	69.3	29.625	69.8	60	62.7	64.0	58.7	69.0	0.019	SSW	Cloudy—light showers.
♃ 8	29.733	73.7	29.670	68.2	46	62.3	65.0	53.7	67.3	0.036	NW var.	Fine and clear—light clouds and wind.
♀ 9	29.464	66.4	29.524	66.2	47	59.8	60.5	49.7	65.3		WNW	Fair—cloudy.
♄ 10	29.782	67.7	29.857	66.9	50	60.5	63.2	52.3	66.3	0.019	W	Lightly cloudy.
☉ 11	29.836	62.3	29.723	64.3	55	57.7	62.4	53.5	64.7		E	Cloudy—light showers.
☽ 12	29.583	65.7	29.805	67.2	61	62.7	64.8	57.7	66.4	0.056	W	Fine—cloudy—showery.
♂ 13	30.156	72.7	30.164	69.2	50	65.4	71.2	48.8	72.3	0.044	W	Fine and clear—light clouds.
☿ 14	30.190	67.6	30.060	68.8	58	68.5	73.5	54.7	74.5		SSE	Fine—nearly cloudless.
♃ 15	29.994	70.3	29.992	70.7	63	67.4	69.8	58.7	70.8		S	Fine—light clouds.
♀ 16	30.038	74.3	30.037	71.6	53	67.7	68.4	57.6	73.7		SSW	Fine—light clouds.
♄ 17	30.079	74.3	29.972	69.8	56	65.4	64.2	52.8	66.7		SSE var.	A. M. Clear. P. M. Overcast. At night, high wind and rain.
☉ 18	29.824	67.4	29.836	67.7	63	63.7	62.3	61.4	67.3	0.083	SSE	A. M. early, wind and rain.—Continued rain.
☽ 19	29.997	72.8	30.039	70.5	56	66.7	69.2	54.3	70.7	0.333	W	Fine—light clouds—showery.
● ♂ 20	30.143	69.7	30.133	70.5	56	64.7	64.0	55.7	69.0	0.033	SW	Lightly cloudy.
☿ 21	30.228	68.6	30.228	70.6	60	65.2	73.4	58.6	75.7		W	Fine—light clouds.
♃ 22	30.248	66.7	30.209	70.2	60	64.3	70.4	60.4	72.8		W	Overcast.
♀ 23	30.155	74.4	30.129	73.7	61	69.7	73.7	62.3	77.3		SW	Overcast. Fine A. M.
♄ 24	30.090	74.0	30.116	74.2	62	70.3	75.0	64.4	75.7		SW	Fine and clear—light clouds.
☉ 25	30.265	74.3	30.255	75.6	62	69.7	79.8	59.6	81.5		SW	Fine—lightly cloudy.
☽ 26	30.280	79.3	30.246	78.3	56	77.2	81.6	66.3	85.2		S	Fine—lightly cloudy.
♂ 27	30.373	80.3	30.344	78.8	60	76.7	81.8	66.6	83.7		E	Clear and cloudless.
☿ 28	30.362	77.7	30.313	77.6	67	74.5	80.4	60.3	82.6		N	Clear and cloudless.
♃ 29	30.197	78.3	30.117	79.6	68	75.7	80.8	64.3	83.3		NNE	Fine—lightly cloudy.
♀ 30	29.977	78.7	29.911	81.0	69	76.5	82.8	66.8	85.8	0.014	W	Fine and clear—lightly cloudy.
♄ 31	30.065	74.3	30.073	78.7	66	71.7	78.7	66.7	79.7		W	A. M. Overcast. P. M. Fine and clear.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	29.988	71.4	29.975	71.5	58	66.9	70.4	58.1	73.0	1.307		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.890 } { 3 P.M. 29.877 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge ..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about)..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR AUGUST, 1830.

1830. August.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	Fahrenheit. 3 P.M.	Self-registering. Lowest.	Self-registering. Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☉ 1	30.026	74.6	29.951	75.1	55	71.6	75.4	60.6	76.3		SSW	Fine and clear—light breeze. Evening lowering.
☽ 2	29.844	71.8	29.862	74.3	59	67.2	71.7	58.7	72.2		SSW	A.M. Showery. P.M. Fine and clear—light clouds.
♂ 3	30.023	72.4	30.052	72.7	55	67.3	70.5	55.5	71.3	0.281	W	Fine and clear—cloudy. Evening lowering.
☿ 4	30.071	67.8	29.982	73.4	61	64.4	73.7	58.7	75.7		ESE	A.M. Cloudy. P.M. Fine and clear—light clouds.
♃ 5	29.918	71.7	29.963	73.3	58	67.3	72.7	64.3	73.7		W	Fine—lightly cloudy.
♀ 6	30.032	69.7	30.035	70.7	51	63.7	68.3	53.3	68.7		WSW	Clear—cloudy—light wind.
♄ 7	29.948	67.3	29.870	70.3	48	64.6	68.7	55.7	70.5		SSW	Fine—lightly cloudy.
☉ 8	29.723	64.8	29.737	69.8	56	62.7	69.8	57.6	70.7		ESE	Fine—lightly cloudy.
☽ 9	29.842	67.7	29.788	70.8	68	64.7	69.8	56.3	72.3		ESE	Overcast.
♂ 10	29.664	66.4	29.639	69.0	61	62.7	65.2	57.6	66.3		ENE	Overcast.
☿ 11	29.729	65.5	29.714	70.3	60	61.7	71.8	57.8	73.4		SSW	A.M. Hazy. P.M. Fine—at night, rain.
♃ 12	29.770	70.7	29.802	71.1	57	65.8	69.8	57.3	70.7	0.072	SSW	Fine and clear—light clouds and wind.
♀ 13	29.700	66.1	29.687	66.8	60	60.3	63.8	58.6	63.5	0.028	WSW	A.M. Rain. P.M. Fine. Thunder at 3h.
♄ 14	29.892	66.3	29.734	66.8	55	61.3	60.0	53.3	66.7	0.736	WSW	A.M. Fair. P.M. Heavy rain.
☉ 15	29.760	65.4	29.808	68.7	54	59.7	66.7	52.3	69.3	0.597	WNW	Fine and clear—cloudy.
☽ 16	29.957	65.3	29.942	66.7	51	59.3	64.3	48.7	67.7		W	Fine—light haze and clouds.
♂ 17	29.978	63.3	29.955	65.6	49	56.7	61.6	46.8	63.8	0.038	W	Fine and clear—lightly cloudy. Evening, rain.
● ☿ 18	30.155	62.7	30.185	64.3	47	56.3	62.6	46.8	63.6	0.169	N	Fine and clear—light clouds and wind.
♃ 19	30.175	59.7	30.118	63.9	47	56.7	62.3	47.7	65.2	0.014	NNW	Fine and clear—light clouds.
♀ 20	29.975	58.4	29.957	63.7	44	54.7	59.3	47.6	62.0		NNW	Fine—lightly cloudy.
♄ 21	30.014	60.5	29.986	64.0	50	56.0	63.5	50.0	65.5		NNE	A.M. Overcast. P.M. Fine.
☉ 22	30.031	62.0	30.024	64.3	50	58.7	66.7	47.0	68.5		S	Fine—light clouds. At night, light rain.
☽ 23	29.994	64.0	29.958	67.2	57	62.0	65.8	58.0	68.0		NW	Overcast.
♂ 24	29.883	61.4	29.888	62.0	56	61.4	62.8	56.6	70.8		SW	Fine—light clouds. At night, rain.
☿ 25	29.807	65.2	29.770	63.0	56	62.8	65.0	56.6	69.8		WSW	Overcast—light wind—showery.
♃ 26	29.849	64.2	29.873	66.9	55	61.0	64.6	55.8	65.5		SW	Fine and clear—light wind.
♀ 27	29.782	64.6	29.625	67.8	58	60.5	66.0	54.0	69.0		S	A.M. Clear. P.M. Overcast. At night, rain.
♄ 28	29.394	64.4	29.457	66.5	53	58.6	63.5	54.8	64.6	0.250	S	A.M. Overcast—showery. P.M. Clear. Heavy rain at 1h.
☉ 29	29.931	64.5	29.996	66.2	50	59.0	62.8	49.8	65.6	0.500	WSW	A.M. Clear. P.M. Rain. Thunder with rain at 1h. 45m.
☽ 30	30.223	61.5	30.192	64.6	44	50.5	63.4	46.6	64.4		WSW	Fine—light haze.
♂ 31	30.291	59.8	30.261	64.2	48	55.0	66.0	46.0	67.0		S	Fine and clear.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	29.916	65.5	29.897	67.9	53	61.1	66.6	53.9	68.5	2.685		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.834 } { 3 P.M. 29.808 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge ..... =83 feet 2½ in.

..... above the mean level of the Sea (presumed about) ..... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR SEPTEMBER, 1830.

1830. Septem.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	3 P.M.	Self-registering. Lowest.	Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☿ 1	30.342	61.8	30.291	65.8	49	58.5	67.6	50.0	68.8	0.250	SW	Fine and clear.
☽ 2	30.200	62.6	30.085	67.8	52	57.8	67.0	50.6	69.8		WSW	Fine—lightly overcast.
♀ 3	29.835	63.6	29.812	65.8	51	60.5	63.2	53.4	65.4		S var.	Fine—lightly overcast.
♄ 4	29.954	62.2	29.966	65.2	53	58.0	62.8	53.2	65.8		N	Fine and clear—light clouds.
☉ 5	29.704	61.2	29.697	65.2	55	57.4	64.6	54.2	65.2		WNW	Overcast—light rain A.M.
☽ 6	29.562	61.4	29.495	64.9	53	58.2	62.4	52.0	63.6		SW	Overcast. Thunder and lightning, with heavy rain at 4½ P.M.
♂ 7	29.748	61.4	29.779	63.6	53	57.6	59.8	54.0	62.5	0.250	W var.	Fair—lightly overcast.
☿ 8	30.004	61.0	30.079	64.8	53	57.8	63.0	53.6	64.2		SE var.	Fair—lightly overcast.
♃ 9	29.865	60.5	29.792	62.8	50	57.6	59.0	49.2	61.3		S	Overcast. At night, rain.
♀ 10	29.657	59.7	29.701	63.2	56	56.4	61.8	54.0	63.0	0.375	W	Overcast. A.M. early, rain.
♄ 11	29.777	57.5	29.756	61.8	47	53.6	60.2	45.8	61.4		SW	Fine—light clouds.
☉ 12	29.388	59.0	29.396	61.3	50	56.2	58.4	51.0	60.6		S	Fine—light clouds—showers.
☽ 13	29.489	58.2	29.548	61.4	49	54.0	60.0	47.5	62.0	0.250	SW	Fine—light clouds.
♂ 14	29.445	59.6	29.460	62.5	51	57.5	61.5	50.0	65.2		S	Fine—light clouds. A.M. early, rain.
☿ 15	29.663	59.2	29.657	62.2	49	57.5	62.0	49.2	63.4		SE	Fine—lightly overcast. At night, rain.
♃ 16	29.606	61.4	29.710	63.5	54	59.0	63.1	56.5	65.6		WSW	Fine and clear.
● ♀ 17	29.686	58.5	29.641	61.6	53	54.8	59.0	51.5	59.8		E	Overcast. Light rain A.M.
♄ 18	29.594	59.6	29.681	62.0	52	55.8	59.4	52.0	63.5	0.250	SW	Lightly overcast.
☉ 19	29.962	58.4	29.866	62.3	49	55.2	60.2	47.2	63.2		S	A.M. Fine and clear. P.M. Rain.
☽ 20	29.641	60.0	29.609	62.8	50	56.5	61.6	52.0	63.8		SW var.	Fine and clear.
♂ 21	29.200	57.8	29.251	61.3	52	53.0	56.0	51.0	58.2	0.250	ESE	A.M. Rain. P.M. Fine.
☿ 22	29.574	55.2	29.677	59.2	46	50.6	59.5	43.0	60.2	0.250	WSW	Fine and clear.
♃ 23	29.325	57.4	29.410	60.5	52	58.4	60.5	50.0	62.5		S var.	Fine—showery.
♀ 24	29.675	56.3	29.594	58.8	46	53.8	52.6	47.0	57.8		S var.	Nearly cloudless. P.M. Hail with high wind.
♄ 25	29.736	56.0	29.800	58.8	45	53.6	58.5	46.8	59.8	0.250	WSW	Fine and clear—light wind.
☉ 26	30.212	54.0	30.282	57.6	45	50.5	58.5	43.8	60.8		SW	Fine—light clouds.
☽ 27	30.420	54.8	30.380	58.4	47	53.4	59.6	46.5	61.2		SE	Fine—lightly overcast.
♂ 28	30.279	57.6	30.154	59.8	49	57.6	61.8	53.2	62.8		SW	Fair—lightly overcast.
☿ 29	30.037	56.2	30.025	59.0	51	53.2	56.8	50.0	58.2	0.250	S	Fair—lightly overcast.
♃ 30	30.109	53.8	30.016	57.6	47	48.8	55.8	44.0	56.2		SSW	Fair—lightly overcast.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	29.789	58.9	29.785	62.0	50	55.8	60.5	50.1	62.0	2.375		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.734 | 3 P.M. 29.712 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge ..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about) ..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR OCTOBER, 1830.

1830. Oct.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	3 P.M.	Self-registering. Lowest.	Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☿ 1	30.079	54.5	30.071	58.6	46	53.5	58.8	44.2	60.5		S var.	Lightly cloudy.
○ ♄ 2	30.119	57.2	30.049	59.5	49	57.6	59.8	53.4	62.0		SE	Lightly cloudy. Rain at night.
⊙ 3	30.008	59.0	30.034	60.6	56	59.4	59.8	56.2	61.0		S	Overcast—light rain.
☽ 4	30.152	58.9	30.257	61.8	52	56.5	60.2	51.8	62.8		SSW	Fine—light haze.
♂ 5	30.439	55.0	30.411	59.0	45	50.9	57.6	43.8	58.6		S var.	Fine—light clouds.
☿ 6	30.404	55.0	30.344	56.6	47	51.3	53.4	45.2	54.6		WSW	Fair—hazy.
♃ 7	30.385	55.2	30.283	58.4	49	53.5	59.6	48.0	60.8		WSW	Fair—hazy.
♀ 8	30.495	55.6	30.470	57.8	50	52.8	60.2	48.3	61.5		W	Fair—light clouds.
♄ 9	30.538	56.3	30.534	59.3	46	54.8	57.6	52.0	58.2		NW	Overcast—hazy.
⊙ 10	30.547	56.3	30.493	58.8	51	54.2	59.4	50.9	59.4		NNE	Lightly cloudy.
☽ 11	30.411	53.7	30.352	57.5	50	51.4	54.6	45.7	55.3		E	Fine—light haze.
♂ 12	30.366	54.7	30.317	56.8	49	50.7	55.4	49.5	61.2		NNE	Overcast—hazy.
☿ 13	30.404	54.0	30.374	58.1	47	51.2	57.2	46.8	59.5		E	Fine and clear.
♃ 14	30.340	53.3	30.289	56.0	44	49.4	53.8	46.0	55.2		E	Fine and clear.
♀ 15	30.273	50.2	30.190	53.8	38	46.3	53.5	40.2	53.5		ESE	Fine and clear.
● ♄ 16	30.279	49.1	30.277	53.0	41	42.6	53.0	39.0	53.5		NNE	A.M. Fog. P.M. Fine and clear.
⊙ 17	30.374	50.2	30.348	53.0	42	47.8	53.4	41.5	53.2		ESE	A.M. Fog. P.M. Fair—hazy.
☽ 18	30.290	50.2	30.196	53.6	43	48.0	55.2	42.4	56.0		E	Fine and clear.
♂ 19	30.035	52.6	29.994	56.2	46	53.5	59.8	47.4	61.2		ESE	Fair—light clouds.
☿ 20	30.052	56.2	30.051	59.0	51	57.8	62.2	53.8	64.8		S	Fine and clear.
♃ 21	30.210	59.0	30.210	61.5	53	58.5	63.6	54.6	66.0		S	Fine—light clouds.
♀ 22	30.303	60.2	30.264	63.5	55	60.0	65.0	54.8	66.5		S	Fine—brisk wind.
♄ 23	30.418	59.8	30.467	61.2	50	55.2	57.6	53.8	58.6		W var.	Fine—lightly overcast.
⊙ 24	30.487	57.3	30.352	59.5	47	51.6	63.2	49.5	57.0		SW	Cloudy. A.M. Thick haze.
☽ 25	30.083	57.9	29.920	59.8	52	55.8	59.5	51.2	59.6		S	Overcast. Rain A.M.
♂ 26	30.021	55.6	30.148	56.4	45	48.9	50.0	46.2	52.2	0.708	S	Fine—light clouds.
☿ 27	30.372	49.4	30.278	53.5	35	41.0	51.2	35.2	54.5		SW	Fair—hazy. Rain at night.
♃ 28	29.940	54.1	29.886	57.4	49	54.8	59.6	40.2	60.6	0.073	SW	Cloudy. Rain at night.
♀ 29	29.695	53.2	29.671	55.2	43	48.8	52.5	45.5	53.4	0.100	WSW	A.M. Fine and clear. P.M. Cloudy.
♄ 30	30.014	49.2	30.033	51.0	37	42.4	46.8	39.0	51.5		S var.	Fine—light clouds.
⊙ 31	29.916	50.7	29.923	53.8	50	51.7	57.2	41.5	58.3		SW	A.M. Foggy. P.M. Nearly cloudless—light wind.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	30.240	54.6	30.212	57.4	47	52.0	57.1	47.0	58.4	0.881		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 30.186 | 3 P.M. 30.181 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge..... = 83 feet 2½ in.

..... above the mean level of the Sea (presumed about) ..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR NOVEMBER, 1830.

1830. Nov.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	3 P.M.	Self-registering. Lowest.	Highest.	Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☽ 1	30.127	54.3	30.130	56.2	53	53.4	57.0	49.6	57.7		SW	Fine and clear.
♂ 2	30.141	54.0	30.085	56.8	50	51.7	56.5	49.3	57.3		WSW	Overcast.
☿ 3	29.927	54.8	29.843	56.6	52	52.4	55.5	50.7	55.7		S	Fine—light clouds.
♃ 4	29.856	55.3	29.831	56.8	50	49.7	55.6	48.3	55.8		SW	Fine—lightly cloudy. At night, strong wind.
♀ 5	29.961	54.4	29.873	57.0	51	50.8	54.6	47.3	55.7	0.047	SSW	Fine—lightly overcast. At night, strong wind.
♄ 6	29.570	56.7	29.397	58.4	55	55.7	58.4	50.5	58.5	0.008	SSE	Cloudy. High wind through the night.
☉ 7	29.136	56.5	29.246	56.7	51	51.5	51.6	50.8	51.3	0.208		A.M. Rain. P.M. Fine—light clouds.
☽ 8	29.654	49.6	29.732	52.5	45	45.4	49.2	41.7	49.2	0.047	W	Fine—lightly cloudy.
♂ 9	29.902	47.7	29.881	50.4	34	40.4	48.2	34.7	48.7		WSW	Cloudless—light haze.
☿ 10	29.775	50.2	29.681	53.3	46	49.6	53.2	39.3	53.6	0.014	SW	A.M. Cloudless. P.M. Light rain. At night, high brisk wind, with showers.
♃ 11	29.533	52.7	29.542	53.5	48	49.6	50.8	47.8	51.2	0.458	SW	A.M. Fine. P.M. Light rain.
♀ 12	.....	.....	30.055	51.0	...	.....	48.3	.....	.....			Fine—light haze.
♄ 13	29.845	48.9	29.651	51.0	40	46.5	48.9	39.0	50.8		ESE	Fine—light clouds.
☉ 14	29.626	51.2	29.606	53.3	46	50.5	52.8	45.5	53.8		S	Fine—lightly overcast. At night, rain.
● ☽ 15	29.602	50.2	29.655	52.6	44	47.8	51.8	42.2	53.4	0.180	S var.	Fine—light clouds. At night, rain.
♂ 16	29.319	53.2	29.065	54.4	47	53.6	54.8	46.8	54.8		E var.	Rain—brisk wind.
☿ 17	29.483	50.6	29.550	52.5	44	45.5	48.8	41.8	51.6	0.033	S	Fine—light clouds. Hail storm at 2½ P.M., and rain at night.
♃ 18	29.784	47.8	29.875	47.8	39	40.5	45.2	38.5	44.6	0.250	SW	Strong fog.
♀ 19	30.206	46.8	30.210	48.0	35	36.8	42.4	34.6	42.2		SW	Fine—hazy.
♄ 20	30.006	46.4	29.932	48.5	37	43.6	47.2	35.5	47.4		SE	Overcast. Rain P.M.
☉ 21	29.936	47.2	29.893	51.2	41	45.0	50.8	40.2	51.8		S	Fine—lightly overcast. At night, rain.
☽ 22	29.751	50.2	29.889	51.2	45	48.7	48.2	44.0	51.5	0.094	WSW	Fine—light clouds.
♂ 23	30.152	47.5	30.212	49.0	40	43.0	46.8	40.0	47.2		SW	Fine—hazy.
☿ 24	30.418	44.6	30.422	45.1	34	37.8	40.6	35.8	40.4		W	Hazy.
♃ 25	30.386	41.8	30.358	43.8	32	36.0	40.5	33.0	40.5		S	Fine—hazy.
♀ 26	30.126	43.2	29.994	44.5	35	38.6	40.0	34.8	43.0		SE	Overcast.
♄ 27	29.750	42.3	29.671	44.0	35	37.4	39.8	34.8	41.0		SE	Fine—light clouds.
☉ 28	29.493	42.5	29.554	44.8	37	42.0	45.2	36.0	45.2	0·361	SE	Overcast. Light rain P.M.
☽ 29	29.787	44.8	29.835	45.2	40	42.8	43.6	41.2	43.6		ESE	Overcast. Light rain A.M.
○ ♂ 30	29.924	45.4	29.911	46.2	41	43.5	44.4	41.6	44.2		ESE	Overcast—light rain.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	29.833	49.3	29.819	51.1	43	45.8	49.0	41.9	49.7	1.700		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.793 | 3 P.M. 29.774 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge ..... = 88 feet 2½ in.

..... above the mean level of the Sea (presumed about) = 95 feet.

The external Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House ..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR DECEMBER, 1830.

1830. Decem.	9 o'clock, A.M. Barom.	9 o'clock, A.M. Attach. Therm.	3 o'clock, P.M. Barom.	3 o'clock, P.M. Attach. Therm.	Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer. Fahrenheit. 9 A.M.	Fahrenheit. 3 P.M.	Self-registering. Lowest.	Self-registering. Highest.	Rain, in Inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
☿ 1	30.029	45.3	30.010	45.2	38	42.6	42.0	41.0	42.2		ESE	Overcast—light fog.
♃ 2	29.867	44.0	29.740	43.5	36	39.8	37.5	38.3	39.4		ESE	Overcast.
♀ 3	29.602	41.8	29.600	42.4	36	37.8	39.6	35.5	39.2	0.047	ESE	Foggy. Light rain A.M.
♄ 4	29.752	41.8	29.776	42.4	33	38.8	40.0	37.0	39.6		SE	Lightly foggy.
☉ 5	29.756	42.3	29.590	43.6	32	39.8	42.3	38.0	43.4		ESE	Fine—light clouds.
☽ 6	29.118	43.0	29.080	44.5	36	44.5	46.3	35.2	46.5		ESE	Overcast—light rain.
♂ 7	29.150	45.6	29.166	46.5	41	44.6	45.8	43.5	46.6		E	Overcast.
☿ 8	29.201	46.8	29.176	47.0	41	45.6	45.0	44.0	45.0		ESE	Overcast. Light rain P.M.
♃ 9	28.897	46.2	28.869	48.0	39	45.6	47.8	42.5	47.2	0.294	ESE	Overcast. Rain A.M.
♀ 10	29.035	47·4	29.119	47.9	40	43.0	43.3	41.8	43.3	0.186	SW	Light fog.
♄ 11	29.314	43.5	29.307	45.4	33	36.8	41.8	34.4	42.0		SW	Fine—lightly cloudy.
☉ 12	29.404	41.0	29.635	42.7	32	33.5	39.2	31.0	39.2		SW	Fine—lightly cloudy.
☽ 13	30.190	37.2	30.277	38.8	26	31.0	35.2	28.8	35.0		S	Fine—light haze.
♂ 14	30.398	36.5	30.386	38.8	24	32.8	40.0	28.3	40.2		S	Overcast and foggy. Light rain P.M.
● ☿ 15	30.471	38.8	30.382	40.3	31	37.0	39.5	32.0	39.8		SW	Foggy.
♃ 16	30.361	39.2	30.368	39.9	32	35.5	35.2	34.0	36.2	0.136	E	Fine—lightly cloudy. Light rain A.M.
♀ 17	30.025	37.7	29.960	39.6	29	32.6	38.2	27.9	38.0		SW	Fine—light clouds. Snow A.M.
♄ 18	30.162	37.0	30.176	38.5	28	33.6	36.0	31.8	36.0	0.186	N var.	Fine—light clouds.
☉ 19	30.186	37.2	29.971	38.7	29	33.0	37.5	30.0	42.2		SW	Overcast.
☽ 20	29.491	39.8	29.558	41.4	28	39.0	41.8	30.2	41.8		S var.	Fine—light clouds.
♂ 21	29.823	38.7	29.784	40.0	30	35.6	40.0	33.8	44.2		SW	Fine—hazy.
☿ 22	29.060	41.8	29.400	43.8	37	44.9	48.2	34.8	48.3		SW	Light rain.
♃ 23	29.427	40.1	29.459	40.0	31	34.8	33.6	32.0	34.2	0.033	SSW	Hazy. Snow early A.M.
♀ 24	29.439	32.7	29.314	32.4	13	20.4	23.7	18.6	23.7		NW	Fine—lightly cloudy.
♄ 25	29.284	28.6	29.275	29.7	16	18.4	24.5	15.8	24.5		WSW	Lightly overcast and hazy.
☉ 26	29.292	25.8	29.260	29.7	19	22.6	30.0	16.8	30.0		W	Lightly cloudy and hazy.
☽ 27	29.167	29.7	29.165	31.7	28	28.5	31.3	21.7	32.2		E	Light clouds and haze.
♂ 28	29.041	32.4	29.134	33.3	31	33.0	34.2	27.3	34.2		E	A.M. Snow. P.M. Light rain.
○ ☿ 29	29.656	32.6	29.657	34.8	30	31.0	36.3	28.6	36.3		WSW	Fair—light clouds.
♃ 30	29.453	35.6	29.282	37.0	33	36.7	38.2	29.4	45.2		ESE	Overcast and foggy. Light rain P.M.
♀ 31	29.185	40.4	29.274	41.8	42	45.5	44.2	35.7	45.4		S var.	Lightly overcast and cloudy—brisk unsteady wind.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	29.588	39.0	29.589	40.3	31	36.0	38.7	32.2	39.4	0.882		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ..... { 9 A.M. 29.574 | 3 P.M. 29.572 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge..... = 83 feet 2½ in.

.....above the mean level of the Sea (presumed about) ..... = 95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House..... = 79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are gradusted by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.